Probability and Geometric Series
Sample Lab Report

Problem 1. (a) The probability of winning on the first roll is simply the probability of getting either a 1 or a 2 on that roll.

For a single dice roll the sample space has six elements ($S = \{1, 2, 3, 4, 5, 6\}$), all of equal probability 1/6. The event of winning on that roll is the subset $W = \{1, 2\} \subset S$, which has two elements. The probability of that event is thus

$$P(W) = P(X = 1) + P(X = 2) = 1/6 + 1/6 = 1/3$$

(b) On any single dice roll, there are two possibilities – the roller either wins (W) or does not win and passes the turn to the other player (E). The probability of W on any single dice roll was computed already as 1/3; the probability of E on any single dice roll can be computed similarly – the event as a subset of S is $E = \{3, 4, 5, 6\}$, and so its probability is

$$P(E) = P(X = 3)+P(X = 4)+P(X = 5)+P(X = 6) = 1/6+1/6+1/6+1/6 = 2/3$$

In order to win specifically on our second roll of the game, we need for the individual rolls to proceed in the exact order E, E, W. These are unrelated events, and so we can compute the probability of them all happening with the product

$$P(\text{win on our second roll}) = P(E)P(E)P(W) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{27}$$

(c) In order to win on our Nth roll, we need for the individual rolls to proceed in the exact order $E, E, \ldots, E, E, W$, where there are a total of exactly $2N - 2$ of the E's preceding the W (this is because we need to not win on each of our first $N - 1$ rolls, and likewise our opponent). Again these are independent events, and so we can compute the probability of them all happening with the product

$$P(\text{win on our Nth roll}) = P(E)P(E) \cdots P(E)P(E)P(W) = \frac{2}{3} \cdot \frac{2}{3} \cdots \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2^{2N-2}}{3^{2N-1}}$$

(d) We need to compute the sum

$$P(\text{win}) = \sum_{N=1}^{\infty} P(\text{win on our Nth roll}) = \sum_{N=1}^{\infty} \frac{2^{2N-2}}{3^{2N-1}} = \sum_{k=0}^{\infty} \frac{2^{2k}}{3^{2k+1}} = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right) \left(\frac{4}{9}\right)^k$$

This is a geometric series, with first term equal to 1/3 and ratio equal to 4/9. Since this ratio has absolute value less than one, we can apply the formula for the sum of an infinite geometric series to get

$$P(\text{win}) = \frac{a}{1 - r} = \frac{\frac{1}{3}}{1 - \frac{4}{9}} = \frac{3}{5}$$

Problem 2. (a) As computed in the statement of the problem, we have

$$\begin{aligned} U_1 &= D \\ U_2 &= D + pD \\ U_3 &= D + pD + p^2D \end{aligned}$$

This pattern continues each day, giving us

$$U_N = D + pD + p^2D + \ldots + p^{N-1}D$$

(b) As we see from the formula computed above, $U_{N+1} - U_N = p^N D > 0$; therefore $U_{N+1} > U_N$.

(c) Since U_N is given by a geometric series with ratio equal to p, and since $p < 1$ (since p is given to be a fraction of 1), we know that U_N must approach a finite limit. We can add it up directly by first using the finite geometric series formula to get

$$U_N = D\frac{1-p^N}{1-p}$$

We can then compute L as the limit of this expression:

$$L = \lim_{N\to\infty} D\frac{1-p^N}{1-p} = \frac{D}{1-p}$$

(d) Of course it would be very important for the person who decides on dosages to know this formula, because that person would need to ensure the eventual limiting level of the drug in the patients body is the medically desired amount. Without knowing the formula, it would be possible for the patient to have an insufficient level of the drug in his/her system, or to have a level that is too high, to the extent that it could be dangerous to the patient's health.

If $p = 1$, then of course the geometric series does not converge and all of the computations above are invalid. The series then becomes simply

$$D + D + D + D + \ldots$$

The level of the drug in the patient's system would then increase without bound, presumably causing negative health effects.

Of course in this case there would be no need for the patient to take the drug continually; rather, the doctor could simply administer the desired amount in one dosage and it would stay in the patient's system forever.