Math 103 Notes on Linear Algebra

Introduction

Linear algebra is a rich and beautiful field of mathematics, with powerful applications in many fields of study including engineering, computer science, physics and economics. And there is much to be said about linear algebra, most of which we will not have time to cover in this course. A solid introduction to linear algebra is given in Math 104, which many students in this course will take next semester.

Despite this ordering of the courses, great uses can be made of linear algebra in a course on multivariable calculus. In fact, the understanding of even the most basic tools from linear algebra creates the opportunity to organize into a single object many ideas from multivariable calculus that would otherwise seem disconnected. In this discussion, we will introduce those needed tools.

Preliminaries

Notation: We begin with a reminder about notation. In the past students have probably seen the notation $\vec{v} = (v_1, v_2, v_3)$ to indicate a vector in $\mathbb{R}^3$ with coordinates v_1, v_2, and v_3. This notation has the advantage that it fits neatly on a typed line along with text. However in linear algebra it turns out to be more natural to write a vector as a column:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

(Admittedly, the reasons for this are based on an arbitrary convention; however, this convention is uniformly accepted in the field, and so we will use both notations regularly in this document and in this course.)

As we study linear algebra, we will very often find ourselves taking the sum of scalar multiples of vectors; so, we give this construction a name.

Definition: A "linear combination" of the vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is a vector that can be written in the form

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

for some real constants $c_1, \dots, c_k$.

Warning: Vectors have several coordinates, so sometimes we use a subscript to distinguish these coordinates from each other, as we did in the notational comment above. However in other instances we may have an arbitrary collection of vectors, and will use a subscript to distinguish the vectors from each other, as we did in the above definition of linear combinations.

Note that these are completely different uses for subscripts, and by necessity each will be used routinely – so, students must be very careful to notice which role the subscript is playing in any given situation. In order to assist in this, in this course ALL vectors will be written either in bold (as is done in the book), or with an arrow over the top, as in $\vec{v}$ – so for example, we know that $\vec{v}_i$ must represent the ith vector in a list of vectors. Students should be aware however that outside of this course, there may be no such notation to distinguish a vector as such, and so context may be the only indication.

Linear Transformations

Students in this course have already seen many examples of multivariable functions. A multivariable function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function that takes in n input variables, and returns m output variables, each of which might depend on all of the inputs. For example, this could be written in an expanded form as

$$f\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \\ \vdots \\ f_m\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) \end{bmatrix}$$

In general, these individual component functions can be any arbitrary functions.

Here we will discuss a very special category of multivariable functions called linear transformations.

Definition: A "linear transformation" $T : \mathbb{R}^n \to \mathbb{R}^m$ is a multivariable function such that for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, and any real number c, we have:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

(2) $T(c\vec{x}) = cT(\vec{x})$

Without too much trouble one can show that these two conditions are equivalent to the following single condition:

Theorem: A multivariable function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation iff for any vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, and any real numbers c_1, c_2, we have:

(3) $T(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2)$

One can also show that this is equivalent to another single condition:

Theorem: A multivariable function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation iff for any vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, and any real numbers $c_1, \ldots, c_k$, we have:

(4) $T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$

We leave these demonstrations to the interested reader.

Thought of in this last way, we can interpret a linear transformation as being a function that "commutes with linear combinations" – that is, you can apply the linear transformation before or after taking a linear combination, and it makes no difference.

To say this in more detail: given a collection of vectors $\{v_i\}$, constants $\{c_i\}$ (to be used as coefficients in a linear combination), and a linear transformation T, one can first take the linear combination of the vectors and then apply the linear transformation (the left side of the above equation), or first apply the linear transformation to the vectors and then take the linear combination of the images (the right side of the above equation) – and these computations always give the same result.

Example: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ 5x_2 \end{bmatrix}$$

To see if this is a linear transformation, we check that it satisfies the above condition, with the arbitrary vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$:

$$\begin{aligned} f(a\vec{x} + b\vec{y}) &= f\left(a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \\ &= f\left(\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2(ax_1 + by_1) + (ax_2 + by_2) \\ (ax_1 + by_1) - 3(ax_2 + by_2) \\ 5(ax_2 + by_2) \end{bmatrix} \\ &= a\begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ 5x_2 \end{bmatrix} + b\begin{bmatrix} 2y_1 + y_2 \\ y_1 - 3y_2 \\ 5y_2 \end{bmatrix} \\ &= af(\vec{x}) + bf(\vec{y}) \end{aligned}$$

So we see that this function f is indeed a linear transformation.

Example: Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1^2 \\ x_2 \end{bmatrix}$$

It is easy to find a counterexample showing that the needed condition is not satisfied for this function:

$$\begin{aligned} g\left(1\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}1\\0\end{bmatrix}\right) &= \begin{bmatrix}12\\0\end{bmatrix} \\ 1g\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+1g\left(\begin{bmatrix}1\\0\end{bmatrix}\right) &= \begin{bmatrix}6\\0\end{bmatrix} \end{aligned}$$

So, g is not a linear transformation.

Exercise: Show that for any linear transformation T, we must have $T(\vec{0}) = \vec{0}$. *Hint: Make a crafty choice of values for a and b in the condition defining linear transformations.*

Linear Transformations and the Standard Basis Vectors

Recall that the standard basis vectors in $\mathbb{R}^n$ are the n vectors whose coordinates are all 0, except for one coordinate which is 1. These vectors are denoted as "$\vec{e}_i$", where i is the index of the sole nonzero coordinate.

Example: The standard basis vectors in $\mathbb{R}^3$ are

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

There is a very useful connection between linear transformations and the standard basis vectors:

Theorem: If T is a linear transformation and if the images of all of the standard basis vectors ($T(\vec{e}_i)$) are known, then the value of T can be computed for any vector $\vec{x} \in \mathbb{R}^n$.

Proof: We observe that the vector $\vec{x}$ can be written as a linear combination of the standard basis vectors (using the coordinates of $\vec{x}$ as coefficients), and then apply the linearity condition:

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \ldots + x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + \ldots + x_nT(\vec{e}_n) \end{aligned}$$

We can interpret this equation as a prescription for how to accomplish the claim of the theorem; to compute the image of a vector $\vec{x}$, simply compute the linear combination of the known images of the standard basis vectors where the coordinates of $\vec{x}$ are used as the coefficients.

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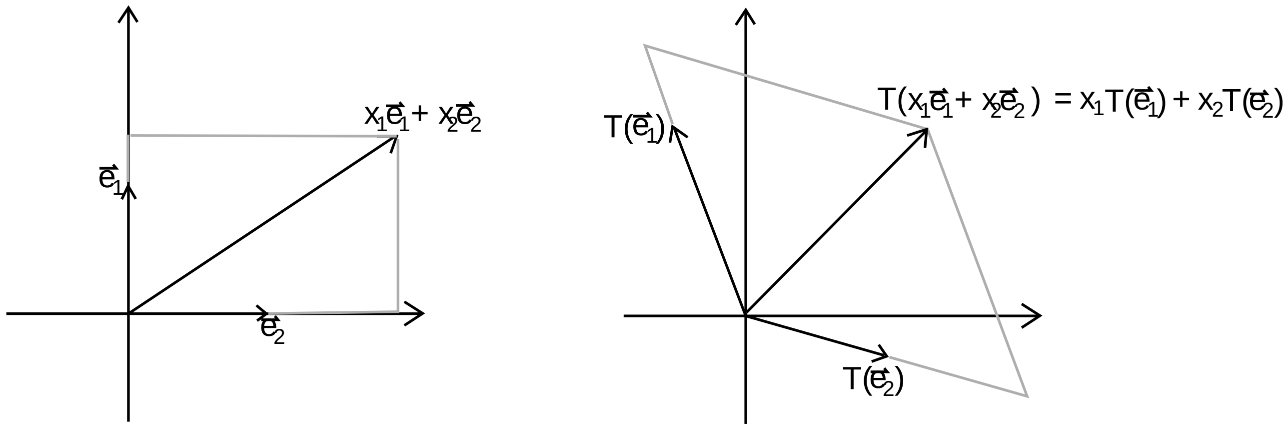
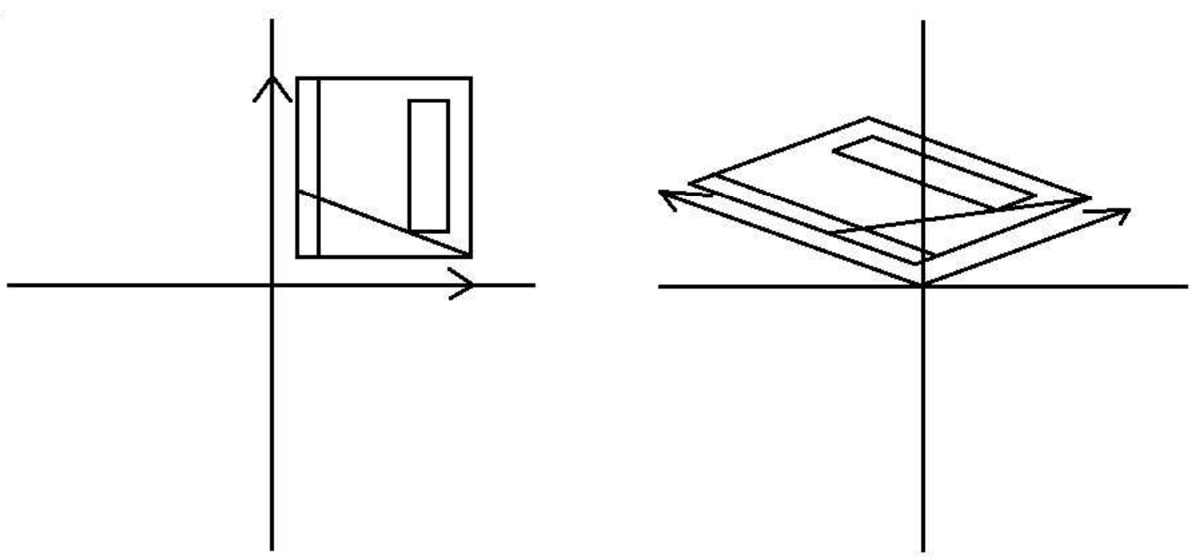


This is a powerful result that allows us to draw strong conclusions about linear transformations. In particular, we can use this to help visualize linear transformations.

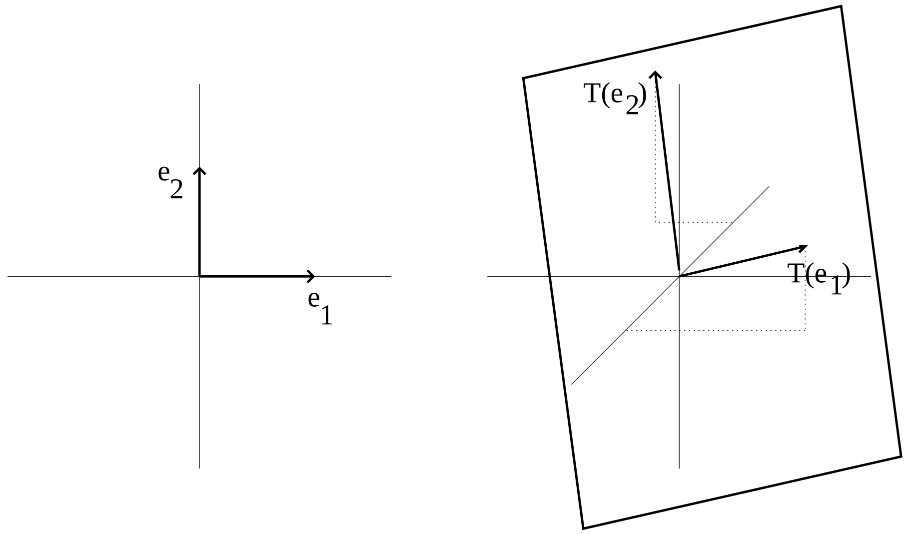
Linear transformations deform the domain in a uniform, linear way, as is shown in the image below depicting a figure in the domain on the left, and on the right the image of that same figure through a linear transformation. Note that even though the image has been stretched and rotated, the relative positions of the objects inside the rectangle are preserved.



Example: Suppose that we have a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, and that we know

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

What is the image of this linear transformation?



We know that every vector in the domain (pictured on the left, above) is a linear combination of the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$; therefore, we know that every image point is a linear combination of the images in the range of those standard basis vectors – which are given (pictured on the right, above). So, the complete image is just the collection of all linear combinations of those image vectors, which is just the plane that is defined by those two vectors. (Sidenote – this is also called the "span" of those image vectors.)

Matrix Notation

As we saw in the first example in the "Linear Tranformations" section, one can represent a linear transformation by giving an explicit formula for evaluating it. But as we just saw with the last theorem, we can also prescribe a linear transformation uniquely by simply listing the images of the standard basis vectors – and then it is understood that we can evaluate the linear transformation on an arbitrary vector by taking linear combinations, as in the theorem.

For example, let's consider the linear transformation from that first example, which we will rename here as T in order to emphasize that it is a linear transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ 5x_2 \end{bmatrix}$$

This formula identifies the linear transformation uniquely, but we could do the same simply by listing

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

with the understanding that the value of T on an arbitrary vector $\vec{x}$ is computed with

$$T(\vec{x}) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

If we make a point of always listing the image vectors in the natural order $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, then we don't even need to identify them individually since it is understood that the first column vector is $T(\vec{e}_1)$, the second is $T(\vec{e}_2)$, ...etc. So we could abbreviate our representation as simply

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \\ 0 & 5 \end{pmatrix}$$

This is called a "matrix". (It is a very common and useful representation of a linear transformation. It should be pointed out here also that matrices have many other uses and interpretations, many of which are equivalent to this one; those other uses and points of view will be discussed in Math 104, but not in this course.)

The reader will note that we gave the matrix the name A, even though it is a representation of the function that has already been named T. The reason for this is that even though we have motivated matrices simply as a shorthand for linear transformations, we will be developing algebraic properties of matrices so that we can think of them as separate objects from the transformations that they represent. This is useful in situations (which will not come up in this course) where matrices are used for other applications. In this course however, even though we write them as distinct things, we can get away with thinking of linear transformations and matrices as being the same thing.

Note that in general there is an immediate relationship between the dimensions of the matrix, and the dimensions of the domain and range of the linear transformation. Suppose we consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$. There are n standard basis vectors in the domain and thus there are n images of standard basis vectors in the range; so the matrix will have n columns. Each of these columns, being an image and thus a vector in the range, will have m components; since each column has m components, this means that the matrix will have m rows.

This matrix with m rows and n columns is referred to as an "$m \times n$" matrix. Notice that as they are written on the page, the m and the n appear here in the opposite order than they appear in the description of the function, $T : \mathbb{R}^n \to \mathbb{R}^m$.

Matrix-Vector Multiplication

As of now a matrix is just a shorthand for a linear transformation, in that its columns represent the images of the standard basis vectors. We have not yet defined any algebraic properties of matrices.

Of course we can make up any definitions we want – but we want to make sure that the algebraic properties we define are natural in some sense that will be useful to us.

For example – we know that we can apply a linear transformation to a vector, and that the result of this is another vector. Since a matrix represents a linear transformation, this suggests that we should define some sort of way that a matrix can be applied to a vector, yielding another vector – such that we get the same thing that we would have had we applied the linear transformation. Since notationally we write a matrix with numbers, we arbitrarily choose to refer to this as "multiplication" of the matrix and the vector.

Specifically then, if A is the matrix representing the linear transformation T, we want to define "matrix-vector multiplication" in such a way that we get

$$T(\vec{x}) = A\vec{x}$$

Before we write this down as a definition, let's denote this motivation as:

Interpretation 0: The matrix-vector product $A\vec{x}$ is the result of the application of the linear transformation T (corresponding to A) to the vector $\vec{x}$.

If we let $\vec{v}_i = T(\vec{e}_i)$ so that

$$A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

then the above equation can be rewritten as

$$\begin{aligned} T(\vec{x}) &= A\vec{x} \\ x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n) &= A\vec{x} \\ x_1\vec{v}_1 + \cdots + x_n\vec{v}_n &= A\vec{x} \end{aligned}$$

We use this as our definition of matrix-vector multiplication.

Definition: Given a matrix A with m rows and n columns, and a vector $\vec{x} \in \mathbb{R}^n$, we define the matrix-vector product $A\vec{x}$ as

$$A\vec{x} = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

The verbal interpretation is:

Interpretion 1: The matrix-vector product $A\vec{x}$ is a linear combination of the columns of A, where the components of $\vec{x}$ are used as the coefficients.

Writing out all of the individual terms of each column vector, and expanding this out and regrouping, it can be shown that we can restate this definition instead in terms of the rows of A; we leave this algebra as an exercise to the interested reader. The result is the following equivalent definition:

Equivalent Definition: Given a matrix A with m rows and n columns, and a vector $\vec{x} \in \mathbb{R}^n$, we define the matrix-vector product $A\vec{x}$ as

$$A\vec{x} = \begin{pmatrix} — & \vec{r}_1 & — \\ & \vdots & \\ — & \vec{r}_m & — \end{pmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

This gives us another interpretation:

Interpretion 2: The matrix-vector product $A\vec{x}$ has as its components the dot products of the rows of A with the vector $\vec{x}$.

Each of these interpretations is important, and will be used extensively in this course.

Matrix Addition and Scalar-Matrix Multiplication

In the last section we developed an algebraic property (which we decided to call "matrix-vector multiplication") that corresponded to the application of a linear transformation to a vector. We are now going to do something very similar to motivate and define two more algebraic properties of matrices – matrix addition and scalar-matrix multiplication.

If we have linear transformations T and S, with matrices A and B respectively, then we can (a) add those transformations, and (b) multiply one of them by a scalar c. In particular, we define the results of those operations by requiring the same results for their actions on vectors. Namely,

Definition:

$$\begin{aligned} (T+S)(\vec{x}) &= T(\vec{x}) + S(\vec{x}) \\ (cT)(\vec{x}) &= cT(\vec{x}) \end{aligned}$$

These new functions are also linear transformations (this is left as an exercise to the interested reader). Very naturally we would want to define our new matrix operations $(A+B)$ and (cA) so that they correspond to these new linear transformations:

Interpretation 0: The sum of two matrices is the matrix corresponding to the sum of the two corresponding linear transformations; the product of a scalar and a matrix is the matrix corresponding to the product of that scalar and the corresponding linear transformation.

Algebraically, this is

$$\begin{aligned}(T+S)(\vec{x}) &= (A+B)\vec{x}\\ (cT)(\vec{x}) &= (cA)\vec{x}\end{aligned}$$

and we can use this setup to compute the individual columns of $(A+B)$ and (cA):

$$\begin{aligned}j\text{th column of } (A+B) &= (A+B)\vec{e}_j = (T+S)(\vec{e}_j) = T(\vec{e}_j) + S(\vec{e}_j) = A\vec{e}_j + B\vec{e}_j\\ &= j\text{th column of } A + j\text{th column of } B\end{aligned}$$

$$\begin{aligned}j\text{th column of } (cA) &= (cA)\vec{e}_j = (cT)(\vec{e}_j) = cT(\vec{e}_j) = c(A\vec{e}_j)\\ &= c(j\text{th column of } A)\end{aligned}$$

Since addition and scalar multiplication of vectors is performed component-wise, this gives us:

Interpretation 1: The sum $(A+B)$ of two matrices A and B is the matrix whose elements are the sums of the corresponding elements of A and B; the scalar-matrix product (cA) is the matrix whose elements are the products of the scalar c with the corresponding elements of A.

We write these out explicitly as our definitions.

Definition: If we write

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

then

$$A+B = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix} \qquad cA = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}$$

Matrix Multiplication

In the last two sections we developed algebraic properties of matrices by analogy with the application of a linear transformation to a vector. We are now going to do something very similar to motivate and define an algebraic property of matrices that naturally corresponds to the composition of linear transformations.

If we have a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ represented by a matrix A, and another linear transformation $S : \mathbb{R}^m \to \mathbb{R}^k$ represented by a matrix B, then we can compose them to form a new linear transformation

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^k$$

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^k$$

This composition transformation can thus be applied to a vector in $\mathbb{R}^n$, and yields a vector in $\mathbb{R}^k$. Notice that because arrows go to the right, while functions are written on the left of the vector they are applied to, the functions are written on the page in the opposite order in these two notations. Students should be especially careful always to be aware of this potential confusion!

By definition of composition of functions, we have

$$(S \circ T)(\vec{x}) = S(T(\vec{x}))$$

If the composition $S \circ T$ is represented by the matrix C, then we can rewrite this in matrix notation as

$$C\vec{x} = B(A\vec{x})$$

Notationally, this is highly suggestive that we should define the matrix C to be the "matrix-matrix product" of the matrices A and B. This gives us

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \quad \Longleftrightarrow \quad (BA)\vec{x} = B(A\vec{x})$$

So as we did with matrix-vector products, before we write down our definition, let's denote our motivation as:

Interpretation 0: The matrix product BA is the matrix that represents the composition of the two linear transformations S and T (represented by B and A, respectively).

There are several ways to represent this algebraically. For each matrix A and B, we will need to refer both to the rows and the columns; here we will use capital letters to denote the rows, and lower case letters to denote the columns:

$$A = \begin{pmatrix} — & \vec{A}_1 & — \\ & \vdots & \\ — & \vec{A}_m & — \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix}$$

$$B = \begin{pmatrix} — & \vec{B}_1 & — \\ & \vdots & \\ — & \vec{B}_k & — \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_m \\ | & & | \end{pmatrix}$$

Let's compute the jth column of the product BA. Using previous ideas, we compute as follows:

$$\begin{aligned} j\text{th column of } BA &= (S \circ T)(\vec{e}_j) \\ &= (BA)\vec{e}_j \\ &= B(A\vec{e}_j) \\ &= B\vec{a}_j \end{aligned}$$

Using this result and our previous interpretations of matrix-vector multiplication, we can make the following interpretations (respectively) of matrix-matrix multiplication:

Interpretion 1: The columns of the matrix product BA are linear combinations of the columns of B, where the components of the corresponding column of A are used as coefficients.

Interpretion 2: An element of the matrix product BA is a dot product of the corresponding row of B with the corresponding column of A.

A third interpretation of matrix multiplication can be shown by explicit expanding and re-grouping from the above; again, those details are left to the interested reader:

Interpretion 3: The rows of the matrix product BA are linear combinations of the rows of A, where the components of the corresponding row of B are used as coefficients.

For convenience, we use interpretation 2 to write down the definition:

Definition: The matrix product

$$BA = \begin{pmatrix} — & \vec{B}_1 & — \\ & \vdots & \\ — & \vec{B}_k & — \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \vec{B}_1 \cdot \vec{a}_1 & \cdots & \vec{B}_1 \cdot \vec{a}_n \\ \vdots & & \vdots \\ \vec{B}_k \cdot \vec{a}_1 & \cdots & \vec{B}_k \cdot \vec{a}_n \end{pmatrix}$$

is the matrix whose element in the ith row and jth column is

$$a_{ij} = \vec{B}_i \cdot \vec{a}_j$$

Example: Compute the matrix product

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$$

We will perform this computation from the points of view of each of the interpretations above.

Interp. 1: We compute the columns individually; the first column is

$$5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 23 \\ 34 \end{bmatrix}$$

and the second column is

$$7 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 31 \\ 46 \end{bmatrix}$$

So the product is

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 23 & 31 \\ 34 & 46 \end{pmatrix}$$

Interp. 2: We compute the individual elements of the matrix by dot products, as

$$\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}\begin{pmatrix}5 & 7\\6 & 8\end{pmatrix} = \begin{pmatrix}\begin{bmatrix}1 & 3\end{bmatrix}\cdot\begin{bmatrix}5\\6\end{bmatrix} & \begin{bmatrix}1 & 3\end{bmatrix}\cdot\begin{bmatrix}7\\8\end{bmatrix}\\ \\ \begin{bmatrix}2 & 4\end{bmatrix}\cdot\begin{bmatrix}5\\6\end{bmatrix} & \begin{bmatrix}2 & 4\end{bmatrix}\cdot\begin{bmatrix}7\\8\end{bmatrix}\end{pmatrix} = \begin{pmatrix}23 & 31\\34 & 46\end{pmatrix}$$

Interp. 3: We compute the rows individually; the first row is

$$1\begin{bmatrix}5 & 7\end{bmatrix} + 3\begin{bmatrix}6 & 8\end{bmatrix} = \begin{bmatrix}23 & 31\end{bmatrix}$$

and the second row is

$$2\begin{bmatrix}5 & 7\end{bmatrix} + 4\begin{bmatrix}6 & 8\end{bmatrix} = \begin{bmatrix}34 & 46\end{bmatrix}$$

So the product is

$$\begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}\begin{pmatrix}5 & 7\\6 & 8\end{pmatrix} = \begin{pmatrix}23 & 31\\34 & 46\end{pmatrix}$$

We have now developed all of the algebraic properties we need for matrices. At this point, anything that we can do with linear transformations we can also do correspondingly with matrices. We can thus think of (for the purposes of this course) linear transformations and matrices as being the same thing.