Math 103 Additional Homework Problems

1. Suppose that the set S in $\mathbb{R}^{n+1}$ is the graph $y = f(\vec{x})$ of the function $f : \mathbb{R}^n \to \mathbb{R}^1$. Find a function $g : \mathbb{R}^{n+1} \to \mathbb{R}^1$ such that S is a level set of g. *(Hint: What can you say about $y - f(\vec{x})$?)*

2. Find a parametric representation of the graph of $y = x^2$.

3. Show that the graph $y = f(x)$ of any single variable function $f : \mathbb{R}^1 \to \mathbb{R}^1$ can be written as a parametric curve.

4. Suppose that the parametric curve $\vec{r}(t) = (f(t), g(t))$ follows the path drawn below; what does the integral

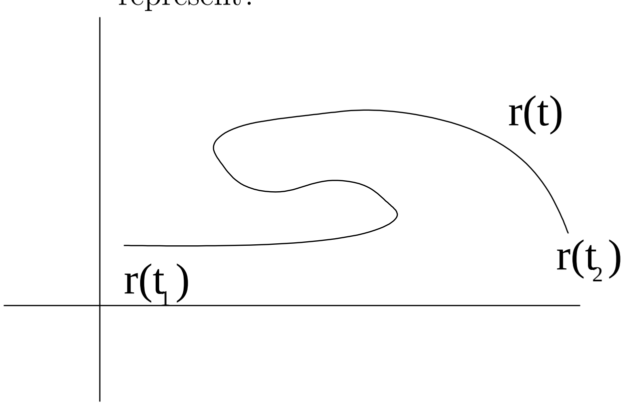
$$\int_{t_1}^{t_2} g(t) f'(t)\, dt$$

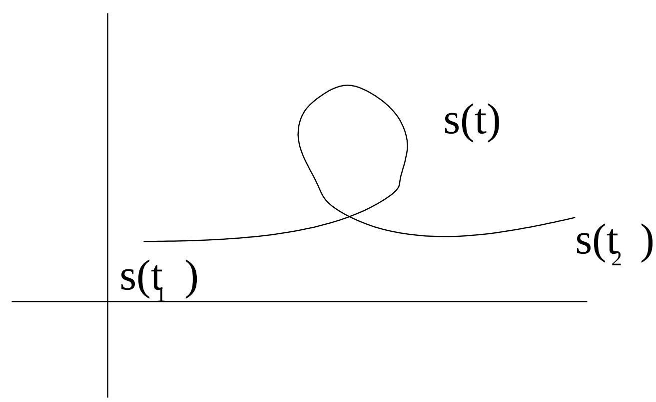
represent?

5. Suppose that the parametric curve $\vec{s}(t) = (h(t), k(t))$ follows the path drawn below; what does the integral

$$\int_{t_1}^{t_2} k(t) h'(t)\, dt$$

represent?





6. Suppose that $\vec{v} = s\vec{u}$. Show, directly from the limit definitions (do not use the gradient formula here!), that

$$D_{\vec{v}} f(\vec{a}) = D_{\vec{u}} f(\vec{a}) \cdot s$$

7. Let $f(x,y,z) = x^2y + xyz$, $\vec{a} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{v} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$. Compute $D_{\vec{v}}f(\vec{a})$ by writing this as a single variable derivative, and computing that single variable derivative directly.

8. Let $f(x,y,z) = x^2y + xyz$, $\vec{a} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\vec{v} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$. Compute $D_{\vec{v}}f(\vec{a})$ by writing this as a product of a scalar and a directional derivative (as in problem 6), and then computing the directional derivative as a single variable derivative (as in problem 7).

9. Show that if $\vec{u} = \frac{\nabla f(\vec{a})}{\|\nabla f(\vec{a})\|}$, then

$$D_{\vec{u}}f(\vec{a}) = \|\nabla f(\vec{a})\|$$

10. Show that the two conditions ((1) and (2)) in the definition of "linear transformation" in Math 103 Notes on Linear Algebra are, as was claimed, equivalent to the single condition (3) in the theorem that immediately follows that definition.

11. Show that the condition (3) demonstrated above is equivalent to the condition (4) in the theorem following it.

12. For each of the functions below, decide if it is a linear transformation and justify your answer based on the definition:

1. $f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x-y\\y-2x\end{bmatrix}$

2. $f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x-y+1\\y-2x\end{bmatrix}$

3. $f\left(\begin{bmatrix}s\\t\end{bmatrix}\right) = \begin{bmatrix}e^c(s+t)\\e^{c^2}t\\c^2s+t\end{bmatrix}$

4. $f\left(\begin{bmatrix}s\\t\end{bmatrix}\right) = s\begin{bmatrix}3\\5\\2\end{bmatrix} + t\begin{bmatrix}-1\\1\\4\end{bmatrix}$

5. $f\left(\begin{bmatrix}s\\t\end{bmatrix}\right) = s\begin{bmatrix}3t\\5s\\2\end{bmatrix} + t\begin{bmatrix}-1s\\1\\4t\end{bmatrix}$

13. Show that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ is a linear transformation if and only if each component function $f_i : \mathbb{R}^n \to \mathbb{R}^1$ is a linear transformation.

14. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and suppose we know that $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Compute the following:

(a) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} 0 \\ 7 \end{bmatrix}\right)$

15. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and suppose we know that $T(\vec{e}_1+\vec{e}_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T(\vec{e}_1 - \vec{e}_2) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. Compute the following:

(a) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

(b) $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

(c) $T\left(\begin{bmatrix} 4 \\ 7 \end{bmatrix}\right)$

16. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and suppose we know only that the image of the unit square (the square with corners at (0,0), (0,1), (1,0), and (1,1)) is the unit square. There are then two possibilities of what T might be – describe the action on the plane of each of these possibilities.

17. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$. Suppose also that $\vec{v}_1$ and $\vec{v}_2$ are non-zero, non-parallel vectors in $\mathbb{R}^2$, and that their images by T are in the xy-plane in $\mathbb{R}^3$. Find a vector $\vec{w}$ in $\mathbb{R}^3$ that is NOT in the image of T, and explain how you know it cannot be in the image of T.

18. A "rigid motion on $\mathbb{R}^2$" is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that "preserves distances" – said precisely, this means that for any vectors $\vec{x}$ and $\vec{y}$, the distance from $\vec{x}$ to $\vec{y}$ is the same as the distance from $f(\vec{x})$ to $f(\vec{y})$. It can be shown that if f is a rigid motion with $f(\vec{0}) = \vec{0}$, then f is a linear transformation.

1. Suppose f is the rigid motion that rotates vectors around the origin counterclockwise by an angle θ. Find the component functions f_1 and f_2 of $f(x, y) = (f_1, f_2)$. *(Hint: First compute the images of the standard basis vectors.)*

2. Suppose g is the rigid motion that rotates vectors around the origin counterclockwise by an angle θ, and then reflects over the x-axis. Find the component functions g_1 and g_2 of $g(x, y) = (g_1, g_2)$.

19. Show that for any linear transformation, we must have $T(\vec{0}) = \vec{0}$.

20. Show that the sum $T+S$ of two linear transformations, defined as $(T+S)(\vec{x}) = T(\vec{x})+S(\vec{x})$, is a linear transformation.

21. Show that a scalar multiple cT of a linear transformation, defined as $(cT)(\vec{x}) = cT(\vec{x})$, is a linear transformation.

22. Show that the composition $S \circ T$ of two linear transformations is a linear transformation.

23. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^4$ is the linear transformation with $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$.

What is the matrix representation for T?

24. In Math 103 Notes on Linear Algebra, the definition of matrix-vector multiplication is followed by an "Equivalent Definition". Writing the matrix A and the vector $\vec{x}$ as

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

compute the matrix-vector product according to each of these formulations, and thus show that this equivalent definition actually is equivalent to the stated original definition.

25. Given the following matrices and vectors, compute the expressions below. Perform the computations in the order that is suggested by the parentheses *(Note: In each grouping, all of the expressions have the same value.)*:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 2 & -1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & 3 \\ 0 & 3 & 1 \end{pmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

1. $A(\vec{x} + \vec{y}), \quad (A\vec{x}) + (A\vec{y})$
2. $A(3\vec{x}), \quad (3A)\vec{x}, \quad 3(A\vec{x})$
3. $(A + B)\vec{x}, \quad (A\vec{x}) + (B\vec{x})$
4. $A(B\vec{x}), \quad (AB)\vec{x}$

26. Find the matrices for each of the linear transformations, based on the given description of how it acts on a vector, by determining the images of the standard basis vectors:

1. $K : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors over the x-axis.

2. $L : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counterclockwise around the origin by $\pi/4$.

3. $M : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors over the x-axis, and then rotates counterclockwise around the origin by $\pi/4$.

4. $N : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counterclockwise around the origin by $\pi/4$, and then reflects vectors over the x-axis

5. $Q : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counterclockwise around the origin by $\pi/4$, and then scales by a factor of 2.

6. $R : \mathbb{R}^2 \to \mathbb{R}^2$ scales vectors by a factor of 2, and then rotates counterclockwise around the origin by $\pi/4$.

7. $S : \mathbb{R}^3 \to \mathbb{R}^3$ rotates vectors by an angle of $\pi/2$ around the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$, in the "right-handed" direction (pointing your right thumb in the direction of the vector, your fingers curl around in the right-handed direction.)

8. $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotates vectors by an angle of $\pi/2$ around the vector $\begin{bmatrix}1\\1\\0\end{bmatrix}$, in the right-handed direction.

27. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is of the form

$$\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}$$

is called a "vertical shear". Draw pictures of the images of the unit square and the unit circle by vertical shears with $k = 1, -1, 2$.

28. What is the first column of the product matrix below? *(Hint: Think about images of standard basis vectors.)*

$$\begin{pmatrix}-3 & 10\\ -5 & 12\end{pmatrix}\begin{pmatrix}3 & 2\\ 5 & -9\end{pmatrix}\begin{pmatrix}-6 & 8\\ 12 & 10\end{pmatrix}\begin{pmatrix}0 & 3\\ 0 & 1\end{pmatrix}$$

29. Using the notation below for the 2x2 matrices A and B, show that Interpretations 1, 2, and 3 from Math 103 Notes on Linear Algebra describing how to perform the matrix multiplication BA are all equal.

$$A = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \qquad B = \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$$

30. Suppose that a non-vertical line in $\mathbb{R}^2$ is the graph of a function $L(x)$, and that this line passes through the point $(a, f(a))$ for some function f. Beginning with the normal vector equation for such a line, show that the function L must be of the form

$$L(x) = f(a) + k(x - a)$$

31. Suppose that a non-vertical plane in $\mathbb{R}^3$ is the graph of a function $P(\vec{x}) : \mathbb{R}^2 \to \mathbb{R}^1$, and that this line passes through the point $(a, b, f(a, b))$ (which we will abbreviate as $(\vec{a}, f(\vec{a}))$) for some function $f(\vec{x}) : \mathbb{R}^2 \to \mathbb{R}^1$. Beginning with the normal vector equation for such a plane, show that the function P must be of the form

$$P(\vec{x}) = f(\vec{a}) + \vec{k} \cdot (\vec{x} - \vec{a})$$

32. Suppose that a non-vertical plane in $\mathbb{R}^{n+1}$ is the graph of a function $P(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^1$, and that this line passes through the point $(\vec{a}, f(\vec{a}))$ for some function $f(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^1$. Beginning with the normal vector equation for such a plane, show that the function P must be of the form

$$P(\vec{x}) = f(\vec{a}) + \vec{k} \cdot (\vec{x} - \vec{a})$$

33. Suppose that the graph of $f(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^1$ is a plane in $\mathbb{R}^{n+1}$, so that we know right off that the tangent plane is in fact just the plane itself, each then being of the form

$$P(\vec{x}) = f(\vec{x}) = f(\vec{a}) + \vec{k} \cdot (\vec{x} - \vec{a})$$

Show by direct computation of the partial derivatives of f with respect to the individual variables $\vec{x}_i$ that in this instance, we must have $\nabla f = \vec{k}$.

34. Show that the function $f(x, y) = x^2 + y^2$ is differentiable at the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, by showing that the limit of the relative error is equal to zero when you use the vector $\vec{k} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. In other words, show that

$$\lim_{\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \frac{\left| f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) - f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - \vec{k} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \right|}{\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|} = 0$$

35. Compute the Jacobian matrix $J_{f,\vec{x}}$ for each of the following functions:

1. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 y \\ y^2 z \\ z^2 x \end{bmatrix}$
2. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} e^x y^2 z \\ z^{xy} \end{bmatrix}$
3. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x^y \\ y^3 \end{bmatrix}$

4. $f(t) = \begin{bmatrix} t^2 \\ e^t \\ \sin t \end{bmatrix}$

5. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x^2y^3z$

36. Evaluate the Jacobian matrix $J_{f,\vec{a}}$ for each of the above functions at the corresponding point below:

1. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} e \\ 2 \end{bmatrix}$

4. π

5. $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

37. The point $\vec{x}$ is at $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and moving with velocity $\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. What is $\frac{df}{dt}$ if $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2y \\ y^2z \\ z^2x \end{bmatrix}$?

38. Use the results from the previous problems to estimate the value of $\sqrt[10]{e}(2.01)^2(2.03)$.

39. Use the results of today's lecture to prove the following theorem, which we asserted in our earlier discussion of vector derivatives:

Theorem: If f is differentiable at the point $\vec{a}$, then

$$D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

40. The cost of a project being planned depends on the choices of three variables – height, area, and density.

Your coworker Ally finds a variation of the original plan that involves increasing the height by 2, decreasing the area by 1, and decreasing the density by 1; the boss notes that this will lower the cost of the project by 1.2 million dollars from the original cost.

Your coworker Ben finds a different variation of the original plan that involves decreasing the height by 1, increasing the area by 2, and decreasing the density by 1; the boss notes that this plan will lower the cost of the project by .5 million dollars from the original cost.

Later you find a variation of the original plan that involves increasing the height by 1, increasing the area by 1, and decreasing the density by 2. Make an estimate of how much your plan will save the company, compared to the original cost.

41. Use the Java Applet called "3D Graph" linked to from our class webpage to create plot of the function $f(x,y) = \frac{3x^2y-y^3}{x^2+y^2}$ (Type ``(3*x^2*y - y^3)/(x^2 + y^2)'' for the function, have xmin and ymin both be -2, have xmax and ymax both be 2, and have xsteps and ysteps both be 41; then click on "Plot Graph".). Moving the slider bar for theta rotates your point of view around the (not drawn) z-axis, and moving the slider bar for phi increases or decreases the angle of your view from the z-axis.

Convince yourself from this inspection that if you take a vertical cross section of this graph through the origin, at any angle, the result is just a straight line; explain why this shows that all of the directional derivatives exist.

Explain in words also, using the graph, why it is that this function is not vector linear.

42. Use the angle addition formulas to show that

$$f(x,y) = \frac{3x^2y - y^3}{x^2 + y^2} = r\sin(3\theta)$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

43. We have shown that the function $g(\vec{x}) = \frac{\|\vec{x}\|^2}{\Theta(\vec{x})}$, with $g(\vec{0}) = 0$, (discussed in class) is not differentiable at $\vec{0}$, even though all of the vector derivatives do exist at $\vec{0}$. Show further that this function is not even continuous at $\vec{0}$.

44. In this problem you will need to use the following fact: If there are n vectors in $\mathbb{R}^m$, and if $m < n$, then those vectors must be linearly dependent (meaning – there must be some linear combination of those vectors which equals the zero vector, where not all of the coefficients in that linear combination are zero).

Given that fact, prove this statement: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with $m < n$, then there must exist some nonzero vector $\vec{x} \in \mathbb{R}^n$ with $T(\vec{x}) = \vec{0}$. *(Hint: Apply the above fact to the images of the standard basis vectors.)*

45. Suppose we have differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$, so that $(g \circ f) : \mathbb{R}^n \to \mathbb{R}^n$.

Show that if $m < n$, then for every point $\vec{a} \in \mathbb{R}^n$ there must be a nonzero vector $\vec{x}$ with $D_{(g\circ f,\vec{a})}(\vec{x}) = \vec{0}$. *(Hint: You can use the Chain Rule to make use of the result of the previous problem.)*

Show by example that if $m \geq n$, then such a vector $\vec{x}$ need not exist. *(Hint: Consider the identity function.)*

46. Suppose we have $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 9x^2y \\ 3xyz \\ 2xz^2 \end{bmatrix}$ and $g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} e^{x-y} \\ e^{y-z} \\ e^{z-x} \end{bmatrix}$. Compute $J_{(g\circ f),(1,2,3)}$.

47. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$$

(Note, the geometric interpretation of this function is that it takes vectors, rotates them counterclockwise around the origin by an angle of $\pi/4$, and then multiplies their lengths by $\sqrt{2}$; you might want to verify this for a few easy vectors to convince yourself this is true...)

Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

1. What is the equation of the parametric curve $\vec{x}(t)$ that moves in a straight line with constant speed, with $\vec{x}(0) = \vec{a}$ and $\vec{x}'(0) = \vec{v}$? Draw a picture representing the path of that parametric curve.
2. What is the equation of the parametric curve $f(\vec{x}(t))$ that is the image of the curve in the previous part? Draw a picture representing the path of that image curve.
3. What is the velocity vector for the image curve when $t = 0$?
4. Determine $D_{\vec{v}}f(\vec{a})$, using your work from the previous parts of this problem.

48. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$$

(Note, the geometric interpretation of this function is that it takes vectors, doubles the angle between that vector and the positive part of the x-axis, and then squares its length; you might want to verify this for a few easy vectors to convince yourself this is true...)

Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

1. What is the equation of the parametric curve $\vec{x}(t)$ that moves in a straight line with constant speed, with $\vec{x}(0) = \vec{a}$ and $\vec{x}'(0) = \vec{v}$? Draw a picture representing the path of that parametric curve.

2. What is the equation of the parametric curve $f(\vec{x}(t))$ that is the image of the curve in the previous part? Draw a picture representing the path of that image curve. (You might want to use the java applet called Parametric Curves that is linked from our class webpage.)

3. What is the velocity vector for the image curve when $t = 0$?

4. Determine $D_{\vec{v}} f(\vec{a})$, using your work from the previous parts of this problem.

49. Suppose that a bee is flying through space in a straight line; at the moment that he passes the point $(3, 2, 4)$ his velocity is $(2, 5, 1)$.

You are observing the bee through a TV monitor, but the camera is not pointed directly at the bee; rather, it is pointed at the image of the bee in a large funhouse mirror. The end result of this is that a point (x, y, z) in space appears on the TV monitor at the location (xe^y, ye^z), measured in horizontal/vertical millimeters from the center of the monitor.

What is the velocity of the bee's image on the monitor at the moment that the bee passes the point $(3, 2, 4)$?