EXAM 2

Math 102, Spring 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. (a) Let $f(x,y,z) = \begin{bmatrix} \ln\left(\frac{x^2+y^2+z^2}{3}\right) \\ 2xy^2z - x^2y \end{bmatrix}$. Use the total derivative of f to estimate the value $f(1.01, .97, .99)$.

Let $\vec{x} = (1,1,1)$, $d\vec{x} = (.01, -.03, -.01)$

$$df = (Df)\,d\vec{x} = \begin{pmatrix} \frac{3}{x^2+y^2+z^2}\,2x & \frac{3}{x^2+y^2+z^2}\,2y & \frac{3}{x^2+y^2+z^2}\,2z \\ 2y^2z - 2xy & 4xyz & 2xy^2 \end{pmatrix} \begin{pmatrix} .01 \\ -.03 \\ -.01 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 2 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} .01 \\ -.03 \\ -.01 \end{pmatrix} = \begin{pmatrix} -.06 \\ -.14 \end{pmatrix}$$

$$f(\vec{x}+d\vec{x}) \cong f(\vec{x}) + df = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -.06 \\ -.14 \end{pmatrix} = \boxed{\begin{pmatrix} -.06 \\ .86 \end{pmatrix}}$$

(b) The function $f : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $f(\vec{x}) = A\vec{x}$, with

$$A = \begin{pmatrix} 2 & 3 \\ 7 & 1 \\ 5 & 0 \end{pmatrix}$$

Compute the derivative matrix Df.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 7x+y \\ 5x \end{pmatrix}$$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 3 \\ 7 & 1 \\ 5 & 0 \end{pmatrix}} = A$$

2. (a) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$, with $f(3,1,2) = (3,1,2)$, and

$$Df(3,1,2) = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 1 & 5 & 4 \end{pmatrix}$$

If $g = f \circ f$, and $g(x,y,z) = (p,q,r)$, compute $\frac{\partial r}{\partial y}(3,1,2)$. *(Make sure to explain how the value of $f(3,1,2)$ is used in this computation!).*

$$Dg(3,1,2) = \left(Df(f(3,1,2))\right)\left(Df(3,1,2)\right)$$

$$= \left(Df(3,1,2)\right)\left(Df(3,1,2)\right)$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \frac{\partial r}{\partial y} & \cdot \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 1 & 5 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 1 & 5 & 4 \end{pmatrix}$$

So $$\frac{\partial r}{\partial y}(3,1,2) = \begin{pmatrix} 1 & 5 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \boxed{39}$$

(b) The production level P of a company is known to depend on two factors, A and B, as given by $P = 5A^{1/2}B^{1/2}$. At a time when $A = 16$ and $B = 4$ we know that $\frac{dP}{dt} = 10$ and $\frac{dA}{dt} = 2$. What is $\frac{dB}{dt}$?

By the chain rule,

$$\frac{dP}{dt} = \frac{\partial P}{\partial A}\frac{dA}{dt} + \frac{\partial P}{\partial B}\frac{dB}{dt}$$

$$\frac{dP}{dt} = \left(5\frac{1}{2A^{1/2}}B^{1/2}\right)\frac{dA}{dt} + \left(5A^{1/2}\frac{1}{2B^{1/2}}\right)\frac{dB}{dt}$$

At the given instant then,

$$10 = \left(5\cdot\frac{1}{2\cdot 16^{1/2}}\cdot 4^{1/2}\right)2 + \left(5\cdot 16^{1/2}\cdot\frac{1}{2\cdot 4^{1/2}}\right)\frac{dB}{dt}$$

$$10 = \frac{5}{2} + 5\frac{dB}{dt}$$

$$\frac{dB}{dt} = \boxed{\frac{3}{2}}$$

3. (a) What is the rate of change of the function $f(x, y, z) = x^2 - y^2z - xz^2$ at the point $\vec{a} = (1, 1, 1)$ when moving at unit speed in the direction of the vector $(6, 2, 3)$?

$$\vec{v} = (6,2,3), \quad \|\vec{v}\| = \sqrt{6^2+2^2+3^2} = 7$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(6,2,3)}{7}$$

$$\frac{df}{dt} = D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$

$$= \begin{pmatrix} 2x - z^2 \\ -2yz \\ -y^2 - 2xz \end{pmatrix} \cdot \begin{pmatrix} 6/7 \\ 2/7 \\ 3/7 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6/7 \\ 2/7 \\ 3/7 \end{pmatrix} = \boxed{-1}$$

(b) In what (unit vector) direction from the point $\vec{a} = (1, 1, 1)$ is the function $f(x, y, z) = x^2 - y^2z - xz^2$ decreasing the most quickly?

Ideal direction is $\nabla f / \|\nabla f\|$, so f decreasing most quickly in direction $-\nabla f / \|\nabla f\|$.

From above, $\nabla f(\vec{a}) = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, so we have

$$\frac{-\nabla f}{\|\nabla f\|} = \boxed{\frac{\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{14}}}$$

4. Suppose we know that $x^2yz + xy^2 - z^2 + 1 = 0$. At the point $(0,2,1)$ on this surface, which of the variables can be expressed in terms of the other two?

$$F(x,y,z) = x^2yz + xy^2 - z^2 + 1 = 0$$

$$\frac{\partial F}{\partial x} = 2xyz + y^2 \quad @(0,2,1), \quad = 4$$

$$\frac{\partial F}{\partial y} = x^2z + 2xy \quad @(0,2,1), \quad = 0$$

$$\frac{\partial F}{\partial z} = x^2y - 2z \quad @(0,2,1), \quad = -2$$

So: x can be thought of as a fn of y, z

y cannot . . . . x, z

z can . . . . x, y

5. Consider the system of equations

$$\begin{aligned} wxy - xyz + 3wz &= 4 \\ w^2 + z^2 + xy &= 7 \end{aligned}$$

near the point $(w, x, y, z) = (1, 2, 1, 2)$.

Show that we can locally view y and z as functions of w and x, and compute $\frac{\partial z}{\partial w}$.

$$D_{yz}\vec{F} = \frac{\partial(F_1, F_2)}{\partial(y, z)} = \begin{pmatrix} wx - xz & -xy + 3w \\ x & 2z \end{pmatrix}$$

$$D_{yz}\vec{F}(1,2,1,2) = \begin{pmatrix} -2 & 1 \\ 2 & 4 \end{pmatrix}$$

$\det D_{yz}F(1,2,1,2) = -10 \neq 0$, so I.F.T. says we can view y, z as implicit functions of w, x.

Diff. system w.r.t. w, we get

$$\left(D_{yz}F\right)\begin{pmatrix} \partial y/\partial w \\ \partial z/\partial w \end{pmatrix} = -\begin{pmatrix} \partial F_1/\partial w \\ \partial F_2/\partial w \end{pmatrix} = -\begin{pmatrix} xy + 3z \\ 2w \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}$$

Cramer's rule then gives us

$$\frac{\partial z}{\partial w} = \frac{\det\begin{pmatrix} -2 & -8 \\ 2 & -2 \end{pmatrix}}{\det\begin{pmatrix} -2 & 1 \\ 2 & 4 \end{pmatrix}} = \frac{20}{-10} = \boxed{-2}$$