EXAM 1

Math 102, Spring 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name ________________________________

ID number______________________

1. ____________ (/20 points)

2. ____________ (/20 points)

3. ____________ (/20 points)

4. ____________ (/20 points)

5. ____________ (/20 points)

Total ____________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. Find the complete set of solutions to the system of equations

$$\begin{array}{rcrcrcr} 1x & + & 2y & - & z & = & 2 \\ 1x & + & 3y & + & 1z & = & 10 \\ 2x & + & 4y & & & = & 10 \end{array}$$

2. In this problem, let

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 9 \\ 8 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 6 \end{pmatrix}$$

Compute the following:

(a) $\vec{v} \cdot \vec{w}$

(b) $\|\vec{v}\|$ and $\|\vec{w}\|$

(c) $\cos(\theta)$ (the angle between $\vec{v}$ and $\vec{w}$)

(d) $A\vec{v}$

(e) AB

3. The work below shows a series of row operations applied to an augmented matrix, whose coefficient matrix we will call A. The notes to the right of the augmented matrix indicate the row operations that were used in that step. Use information from the computation below to help you answer the questions on the following two pages.

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 1 & 0 & 1 & 1 & 2 & b_3 \\ 2 & 10 & 32 & -1 & 5 & b_4 \\ 3 & 0 & 3 & -2 & 9 & b_5 \end{array}\right)$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 10 & 30 & -1 & 1 & -2b_1 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3)\text{-}1(1) \\ (4)\text{-}2(1) \\ (5)\text{-}3(1) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & -1 & 1 & -2b_1 - 2b_2 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) \\ (4)\text{-}2(2) \\ (5) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & -1 & 1 & -2b_1 - 2b_2 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2)/5 \\ (3) \\ (4) \\ (5) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & -3b_1 - 2b_2 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & 3 & -5b_1 + 2b_3 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) \\ (4)\text{+}1(3) \\ (5)\text{+}2(3) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 0 & 7b_1 + 4b_2 - 2b_3 - 2b_4 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & -3b_1 - 2b_2 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & 0 & 4b_1 + 6b_2 - b_3 - 3b_4 + b_5 \end{array}\right) \begin{array}{l} (1)\text{-}2(4) \\ (2) \\ (3) \\ (4) \\ (5)\text{-}3(4) \end{array}$$

(a) Find the complete solution to the system of equations $A\vec{x} = \vec{c}$, where $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\vec{c} = (1, 0, 5, 1, 4)$.

(b) Compute the determinant of A.

(c) Find the precise condition(s) on the components of the vector $\vec{c} = (c_1, \ldots, c_5)$ that will guarantee that the system $A\vec{x} = \vec{c}$ has a solution.

(d) Are there any cases in which $A\vec{x} = \vec{c}$ has a unique solution? (Make sure to explain your reasoning!)

4. Suppose that a series of row operations reduces the augmented matrix representing the system $A\vec{x} = \vec{b}$ (where $\vec{b} = (b_1, b_2, b_3, b_4)$) to the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1b_1 + 3b_2 + 5b_3 + 1b_4 \\ 0 & 1 & 0 & 0 & 3b_1 + 7b_2 + 2b_3 + 1b_4 \\ 0 & 0 & 1 & 0 & 2b_1 + 1b_2 + 6b_3 + 5b_4 \\ 0 & 0 & 0 & 1 & 5b_1 + 1b_2 + 7b_3 + 8b_4 \end{array}\right)$$

(a) What is the solution to the system $A\vec{x} = (0, 0, 1, 0)$?

(b) The solution to the first part of this problem can be related to one of the columns of A^{-1}, by noting the equivalence

$$A\vec{x} = \vec{e}_3 \quad \Longleftrightarrow \quad \vec{x} = A^{-1}\vec{e}_3$$

and also that for any matrix M, we have

$$M\vec{e}_i = i\text{th column of } M$$

Use this idea to compute the entire matrix A^{-1}.

5. The plane P passes through the points $(2, 0, 0)$, $(0, 6, 0)$ and $(0, 0, 3)$ in xyz-space. Let L be the line that is the intersection of P with the plane $y = z$.

(a) Find the point-normal equation for the plane P. (There is more than one possible correct equation.)

(b) The plane P is the graph $z = f(x, y)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Find n, m, and an explicit expression for the function f.

(c) The plane P is a level set $g = 0$ for a function $g : \mathbb{R}^p \to \mathbb{R}^q$. Find p, q, and an explicit expression for the function g. (There is more than one possible correct function.)

(d) The line L is represented parametrically by a function $h : \mathbb{R}^r \to \mathbb{R}^s$. Find r, s, and an explicit expression for the function h. (There is more than one possible correct function.)