Net Worth of a Company

Purpose
The purpose of this lab is to show how mathematical modeling can be used to make business decisions.

Preview
We shall investigate models for the net worth of a company based on its initial capitalization and current expenses. We will see that a simple model predicts that the success of the company depends on the amount of initial capital available and on assumptions on the future behavior of the overall economy. A more sophisticated model leads to a differential equation which cannot be solved explicitly by the methods in this course. It can be solved approximately by Euler’s method, and the approximate solution gives information about the best time to launch the company.

A Simple Model: Initial Capitalization. An entrepreneur has an idea for a new company. By taking the idea to banks and other investors, she finds that she can raise an amount of capital, $W_0$. Our money unit will be one thousand dollars, so for example, an initial capitalization of two million dollars would correspond to $W_0 = 2000$. She lets $W(t)$ denote the net worth of her company where $t$ is measured in years. She assumes that, in the absence of expenses, the company’s assets will have a continuous growth rate of 4% (because the national economy grows at approximately 4% a year). She estimates that the company’s expenses will be $200,000 per year. Using these assumptions, she writes down the following initial-value problem:

$$W'(t) = 0.04W(t) - 200, \quad W(0) = W_0$$  \hspace{1cm} (1)

Problem 1.

(a) Explain the terms in the differential equation.

(b) Solve the initial-value problem (1) to find the net worth as a function of time and the initial capitalization.

(c) Graph the solutions in the cases $W_0 = 4000$ and $W_0 = 6000$.

(d) In the case $W_0 = 4000$, how long will it take before the company goes broke?

(e) What initial capitalization is needed to be sure that the net worth will continue to grow?

This model is very simple; it is easy to think of many things that might affect the net worth of a company that have been left out. Nevertheless, one can learn an important lesson from the model. If you expect a fixed growth rate and fixed expenses, then there is an initial amount of capital necessary for the business to succeed.
A Simple Model: Growth Rate. Let’s use the simple model to make another kind of calculation. Suppose that the entrepreneur can raise initial capital of 4000. According to the above model, if she starts the company, it will go broke. However, there is an assumption in the model that the company will have a growth rate of 4% because the overall economy is expected to grow at that rate. What if the overall economy grows at 5% or 6%? Consider the initial-value problem with $W_0 = 4000$, but the growth rate $\mu$ is unspecified.

$$W'(t) = \mu W(t) - 200, \quad W(0) = 4000 \quad (2)$$

Problem 2.

(a) Solve the initial-value problem (2) to find the net worth as a function of time and the growth rate. (Remember that $\mu$ is constant.)

(b) Graph the solutions in the cases $\mu = .04$ and $\mu = .06$.

(c) For which growth rates $\mu$ will the company succeed?

There is another important lesson here. The decision of whether to start this business or not depends on estimates (guesses!) on the future behavior of economic factors (in this case, the growth rate $\mu$) over which the entrepreneur has no control. This is not an unusual situation. Furthermore, political decisions, such as budget compromises in the Congress of the United States, also depend on estimates of future economic behavior since that behavior has a significant effect on tax revenues. These budget decisions are difficult, not just because of differences in social philosophy, but also because the decisions themselves affect the future growth rate. Therefore, good models for how federal policies (including Federal Reserve policy which is not under the control of Congress) affect future growth rates are absolutely necessary for informed political discussion.

Cyclical Growth Rate. Let us imagine that our entrepreneur finds that she can raise initial capital $W_0 = 7000$. As we saw in Problem 1, this is enough for the company to succeed if the average growth rate of the economy is 4%. In fact, the growth rate has not been constant over the past decades, but has varied both higher and lower than 4%. We will make the (overly) simple assumption that the growth rate varies sinusoidally between 1% and 7% with a period of 4 years and that the company is started ($t = 0$) when the growth rate is 4% and moving up. Define

$$\mu(t) = .04 + (.03) \sin \left( \frac{t\pi}{2} \right), \quad (3)$$

and consider the initial-value problem

$$W'(t) = \mu(t)W(t) - 200, \quad W(0) = 7000.$$  

Problem 3.

(a) Draw a graph of the function $\mu(t)$ for $0 \leq t \leq 8$, and explain carefully why this function fits the assumption made above.

(b) Use Euler’s method with step size $\Delta t = .5$ to compute approximately the net worth of the company at the times $t = .5, 1.0, 1.5, \ldots, 3.5, 4$. Plot these points to get an approximate graph of the net worth as a function of time for the first 4 years.
Problem 4.

(a) Assume as above that the growth rate varies sinusoidally between 1% and 7% with a period of 4 years. However, this time assume that the company is started \((t = 0)\) when the growth rate is 4% and moving down. Explain why \(\mu(t)\) is now given by the formula

\[
\mu(t) = .04 + (.03) \sin \left( \frac{t\pi}{2} + \pi \right).
\]

(b) Use Euler’s method with step size \(\Delta t = .5\) to approximate the net worth of the company at the times \(t = .5, 1.0, 1.5, \ldots, 3.5, 4.\) Compare the result to what you got in Problem 3. Does it make a difference when the company is started? How much of a difference?