RESEARCH STATEMENT

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1. Introduction

I am interested in all areas of theoretical and applied probability. Specifically, I have worked on problems on hermitian and non-hermitian random matrices. Lately, my research is on understanding the behavior of random matrices that arise out of sparse graphs. Along with that I also study the large n behavior of statistical physics models on sparse graphs. Recently I have developed a keen interest on problems on percolation. I am also interested to work in problems arising from other branches of science like Biology, Physics, Statistics, Social Networks etc in which probability theory can provide greater understanding.

Below I briefly describe my works so far, and my current research projects.

2. Limiting spectral distribution of random matrices

Denoting $\lambda_j(A_n)$, $j = 1, 2, \ldots, n$, to be the eigenvalues of a $n \times n$ matrix A_n, let L_{A_n} be the empirical spectral distribution (ESD) of A_n, i.e.

$$L_{A_n} := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j(A_n)}. \tag{2.1}$$

Depending on whether the random matrix A_n is hermitian or not, the random probability measure L_{A_n} will be either supported on the real line or on the complex plane.

2.1. **Limiting spectral distribution of non-hermitian random matrices.** Since the seminal work of Wigner [32] there has been a lot of work in understanding the behavior of eigenvalues of different matrix ensembles. A large proportion of these works are for hermitian matrix ensembles. Compared to the hermitian matrix ensembles, the results on the non-hermitian matrix ensembles are very limited. For example, the conjecture about the *circular law* was proved only recently in [36] after a long series of works (see historical references in [13]). Analyzing the spectrum of a non-hermitian matrix is usually much more challenging than its hermitian analogue. One reason being the sensitivity of the eigenvalues under small perturbation. A very small perturbation in non-hermitian matrices can lead to an entirely different limiting spectrum. Moreover the spectrum now being supported on the complex plane, many of the techniques, like the method of moments, the Stieltjes transform, which work very well for hermitian matrices fail in this regime. This calls for new tools to analyze the spectrum of non-hermitian matrices. Girko [20] proposed a general scheme to analyze the ESD of a non-hermitian matrix A_n by relating it to the singular value distribution of $A_n^v := A_n - vI_n$, for $v \in \mathbb{C}$. This approach requires the convergence of the integral of $\log(\cdot)$ under the singular value distribution of A_n^v for all $v \in \mathbb{C}$. Therefore one needs a careful control on the *small* singular values of A_n^v. Performing this careful analysis, when all entries of A_n are i.i.d. zero

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mean unit variance random variables, Tao and Vu were able to prove the circular law conjecture in [36].
However this delicate analysis of the small singular values is not so straightforward in almost all natural ensembles. Thereby very few results are available in non-hermitian regime. For example, no results are known for sparse directed graphs. More specifically, let us consider an oriented d-regular graph on n vertices. That is, each vertex in the graph has d incoming edges and d outgoing edges. It was conjectured very recently in [13] that the ESD of A_n, the adjacency matrix of a uniformly chosen d-regular oriented graph on n vertices, converge to a measure μ_d, for $d \geq 3$, which has a density given by

$$\mu_d(dz) = \frac{1}{\pi} \frac{d^2(d-1)}{(d^2 - |z|^2)^2} \mathbb{I}_{|z| \leq \sqrt{d}}.$$

It can be checked that μ_d is also the *Brown* measure of *free* sum of d *Haar unitary* operators. Motivated by this observation, with Amir Dembo in [8] we consider the sum of d i.i.d. Haar distributed Unitary and Orthogonal matrices, and prove that the ESD converges to μ_d for all $d \geq 2$. To prove our result we adapt the techniques of [23]. In [23] the authors consider the ESD of $U_n T_n V_n$, where U_n, V_n are i.i.d. Haar distributed Unitary matrices, and T_n is real diagonal such that the imaginary part of the Stieltjes transform of T_n is uniformly bounded. Their method do not immediately yield the result in our set-up because of the boundedness assumption. Using [23] we obtain the *Schwinger-Dyson* equation in our set-up, which relates the Stieltjes transform in the case of sum of d Unitary/Orthogonal matrices with that of $(d-1)$. Analyzing this equation, by induction on d, we identify the unbounded regions as a function of d. This analysis together with the help of the results in [30] give the main theorem of [8]. The boundedness assumption in [23] prevents the presence of any atoms in the limiting spectral distribution of T_n. Even the case $T_n = I_n$ can not be considered in [23]. As a result of our analysis of Schwinger-Dyson equation we can improve the bounded assumption, thereby allowing T_n's for which the limit has atoms, unbounded density and singular distributions.

Next the natural extension will be to prove that the limiting law of the ESD of sum of d i.i.d. uniform permutation matrices is again μ_d. The basis of the methods used in [8] is the analysis of the relevant Schwinger-Dyson equation. The *continuity* property of the unitary and the orthogonal groups is crucial to obtain the Schwinger-Dyson equation. Since the permutation group is discrete, finding an analogue of Schwinger-Dyson is a non-trivial and challenging task. Furthermore, one needs to find results analogous to the ones obtained in [30]. With Amir Dembo, Eyal Lubetzky and Ofer Zeitouni [9], we have taken up these tasks. This work is currently in progress.

In another research project [2], I consider sparse directed Erdős-Rényi graph $G(n, p_n)$ with $p_n \gg \frac{\log n}{n}$ but $p_n \ll n^{\alpha-1}$ $(\alpha > 0)$. The objective is to obtain the limit law of the ESD of the adjacency matrices of such sparse graphs. It is believed that in this regime the limit should be the circular law. However only when $p_n = O(n^{\alpha-1})$, extending the ideas of [36], in [33] the circular law conjecture has been proved. Because of *smaller* number of independent random variables extending the ideas of [33] to this more sparse case is not straightforward. With the help of a slightly different approach, I have been able to prove the circular law conjecture for sparse directed Erdős-Rényi graphs when $p_n \geq \frac{(\log n)^\alpha}{n}$ for some α. Currently I am working on the draft.

2.2. **Limiting spectral distribution of hermitian random matrices.** In the hermitian regime many of the problems are motivated by its application in different fields. Lately statisticians are interested in high dimensional inference, and a lot of problems on random matrices arise from

there. When the number of parameters to be estimated also grows to infinity, obtaining a consistent estimator often becomes a difficult task. Estimators which work well in low dimensional set-up often fail to be consistent. For example, it is well known that when the rows of a $n \times p$ matrix X are i.i.d. $N_p(0, I_p)$, p-variate Gaussian with identity covariance matrix, then the ESD of the normalized sample covariance matrix converges to the Marčenko-Pastur law, whenever $p/n \to (0, \infty)$ (see [26]), which is clearly inconsistent for the spectral distribution of the population covariance matrix. In [6], with Arup Bose and Sanchayan Sen we study similar problems in the context of a time series. We show there that the ESD of sample autocovariance matrices is inconsistent for the population spectral distribution. In high dimensional statistical inference the regularization of covariance matrices by banding/tapering or thresholding has received particular interest (c.f. [11, 12, 19]) in order to obtain consistent estimators. Banding and tapering techniques also appear in time series analysis in the context of nonparametric spectral density estimation. Motivated by these we put some appropriate banding structure on the sample autocovariance matrices, and show that under some mild conditions the ESD is a consistent estimator. If the elements of the sample autocovariance matrices are tapered by a kernel, then the same result continue to hold.
Specifically, in [6] we consider the linear process $X = \{X_t\}_{t\in\mathbb{Z}}$, given by

$$X_t = \sum_{k=0}^{\infty} \theta_k \varepsilon_{t-k} \tag{2.2}$$

where $\{\varepsilon_t,\ t \in \mathbb{Z}\}$ is a sequence of i.i.d. zero mean unit variance random variables and $\{\theta_k\}_{k\in\mathbb{Z}}$ are some given constants. For such a stationary process one can define a population autocovariance matrix $\Sigma_n(X)$, where $\Sigma_n(X)(i,j) = \gamma_X(i-j) = cov(X_i, X_j)$. It is well known that for the autocovariance function $\{\gamma_X(k)\}_{k\in\mathbb{Z}}$ there exists a unique distribution, termed as the spectral distribution, such that

$$\gamma_X(h) = \int_0^1 \exp(2\pi i h x) dF_X(x) \quad \text{for all} \quad h. \tag{2.3}$$

Let $\Gamma_n(X)$ be the sample autocovariance matrix, i.e. $\Gamma_n(X)(i,j) = \hat{\gamma}_X(i-j)$ where the sample autocovariance function $\hat{\gamma}_X(k) = n^{-1}\sum_{i=1}^{n-|k|} X_i X_{i+|k|}$. Though $\hat{\gamma}_X(k) \to \gamma_X(k)$ almost surely for each k, it is well known that the largest eigenvalue of $\Sigma_n(X) - \Gamma_n(X)$ does not go to zero (c.f. [27, 34, 35]). Motivated by this we studied the behavior the limit of the ESD of $\Gamma_n(X)$. Using the method of moments argument, under a weak condition on $\{\theta_k\}_{k\in\mathbb{Z}}$, we show that the limit exists and is inconsistent for the population spectral distribution. That is, the limiting ESD of $\Gamma_n(X)$ is different from that of $\Sigma_n(X)$. Our method of moment argument in [6] is based on the scheme introduced by Bryc, Dembo and Jiang in [15]. However a careful approach needs to be taken here because of the inherent strong non-linear dependence among the entries of $\Gamma_n(X)$.
In order find a consistent estimator, we consider the following two types of banded sample autocovariance matrices: for a sequence of positive integers $\{m_n\}$ such that $m_n \to \infty$ and $m_n/n \to \alpha \in [0, \infty)$ as $n \to \infty$, we define $\Gamma_n^{\alpha,I}(X)$ be the $n \times n$ matrix formed out of $\Gamma_n(X)$ by setting $\hat{\gamma}_X(k) = 0$ whenever $|k| \geq m_n$. Further we let $\Gamma_n^{\alpha,II}(X)$ to be the $m_n \times m_n$ principal minor of $\Gamma_n(X)$. In [6, Theorem 2.3] we show that, under some weak conditions on $\{\theta_k\}$ and for any choice of the distribution of $\{\varepsilon_t, t \in \mathbb{Z}\}$ with zero mean and unit variance, the ESD of $\Gamma_n^{\alpha,I}(X)$ and $\Gamma_n^{\alpha,II}(X)$ is a consistent estimator *iff* $\alpha = 0$. Furthermore for a symmetric bounded kernel K, which is continuous at 0 and $K(0) = 1$ we define a matrix $\Gamma_{n,K}^{\alpha}(X)$ such that its $(i,j)^{th}$ entry is $K((i-j)/m)\hat{\gamma}_X(i-j)$, and show that again it is consistent estimator *iff* $\alpha = 0$.

Toeplitz and Hankel matrices are two examples of non-classical random matrix ensembles which have also received importance lately. Because of the additional structures of these matrices results related to the limiting ESD are limited. The existence of the limiting ESD for these two ensembles were only proved in [15]. In [4], with Arup Bose we study the limiting ESD of a class of random matrices which are closely related to Toeplitz and Hankel matrices. In these matrices we scale each entry of the matrices by the square root of the number of times the random variable corresponding to that entry appears in the matrix. Due to this natural scaling we call them balanced Toeplitz and Hankel matrices. Note that for balanced Toeplitz matrix the diagonal are scaled by $\sqrt{n}$, whereas the top right and bottom left entries are not scaled at all. Due to this non-uniform scaling one needs to carefully adapt the method of moments techniques appropriately to apply it here. Verifying the Carleman's condition becomes an extremely challenging calculus problem. We bypass this issue using a truncation technique, and therefore it is not clear whether the limit moments uniquely characterize the distribution.

In an earlier project with Arup Bose [5] we studied the limiting ESD of several band matrices. Our results include banded version of Toeplitz, Hankel, symmetric circulant, reverse circulant and several palindromic matrices. We noted that for symmetric circulant, reverse circulant and palindromic matrices the limit is universal with respect to the banding parameter (c.f. [5, Theorem 1]). Whereas the limiting ESD of Toeplitz and Hankel matrices do see the effect of banding parameter. For $\alpha = 0$, we prove that the limiting ESD for Toeplitz matrices is Gaussian and that for the Hankel matrices is a symmetrized chi distribution. Using the method of moments here we show that all these results are universal with respect to the distribution of the entries of the matrices. After we finished this work, we learnt about two similar independent works [24] and [25]. Assuming the existence of all moments, in [24] the author considers band Toeplitz matrices and proves the results. In [25] the authors consider band Toeplitz and Hankel matrices with entries having all moments, and proves the relevant result. The latter result is based on the representation of Toeplitz and Hankel matrices as linear combinations of backward and forward shift matrices. Our technique of proof in [5] is able to prove all the results in [24, 25] under less assumptions on the entries and contain a larger class of band matrices.

3. Statistical physics models on sparse graphs

Over the last few decades probabilists have become very interested in understanding many fascinating phenomena of statistical mechanics. Remarkable progresses have been made in percolation theory, interacting particle systems, and recently in conformal invariance. Apart from these dynamic fields of research another area which has received great importance and seen many works being done in recent years, is the study of different statistical physics models on large sparse graphs. Among all the statistical physics models, probably the simplest and non-trivial model is the Ising model. Given any finite graph $G_n = (V_n, E_n)$ the Ising measure on it is given by the following probability measure:

$$\nu_n^{\beta,B}(\underline{x}) = \frac{1}{Z_n(\beta, B)} \exp\Big\{\beta \sum_{(i,j)\in E_n} x_i x_j + B \sum_{i\in V_n} x_i\Big\}, \tag{3.1}$$

where $\underline{x} = \{x_i : i \in V_n\}$ with $x_i \in \{-1, +1\}$, and $Z_n(\beta, B)$ is an appropriate normalizing constant (also known as partition function). In statistical physics literature the parameter B is called the external magnetic field parameter, and β is termed as inverse temperature parameter. When $\beta \geq 0$ the model in (3.1) is termed as ferromagnetic and otherwise anti-ferromagnetic. Initially the Ising model was introduced to understand the behavior of materials at different temperature. This lead to the study of this model on lattices. However recently there is an increasing interest in studying this model on non-lattice complex networks (see [29] and the references therein). Study of this model on sparse graphs is also motivated by its applications in combinatorics, computer science, statistical inference and other fields.

It is often very hard to study the Ising measure on large graphs because of its unknown normalizing constant $Z_n(\beta, B)$. Therefore a first step to understand this measure is to analyze the large n limit of the log-partition function, namely to find the limit of $\frac{1}{n}\log Z_n(\beta, B)$. For sparse graphs this task is achieved in [16, 17, 21, 31]. Beyond that, perhaps the most interesting feature of the distribution in (3.1) is its "phase transition" phenomenon. Namely, for a wide class of graphs it is believed that when $B = 0$ and β is larger than the *critical value* then the Ising measure decomposes into convex combination of well-separated *simple* components. For sparse graphs existence of this phenomenon was proven only for a sequence of k-regular *locally tree-like* graphs (c.f. [28]). In a project with Amir Dembo [7], we consider a more general sequence of tree-like graphs, and prove the *universality* of this phenomenon.

More precisely, we say that the graph sequence $\{G_n\}$ is locally tree-like, when for large n the neighborhood of a typical vertex has approximately the law of the neighborhood of the root of a randomly chosen limiting tree. One expects that the marginal distribution of $\nu_n^{\beta,B}(\cdot)$ converges to the marginal distribution on a neighborhood of the root for some Ising Gibbs measure on the limiting tree T. Since for $B = 0$ and β large, there are uncountably many Ising Gibbs measures, the convergence to a particular Gibbs measure is not at all clear, as is the choice of the correct Gibbs measure. In [28], it was shown that for k-regular trees, the plus/minus boundary conditions play a special role. Indeed, it was proved that if G_n's are locally tree-like graphs, which converge *locally weakly* to k-regular trees $\mathsf{T} = \mathsf{T}_k$ then, for any $\beta > 0$ and $B = 0$,

$$\nu_n^{\beta,0}(\cdot) \to \frac{1}{2}\nu_{+,\mathsf{T}}^{\beta,0}(\cdot) + \frac{1}{2}\nu_{-,\mathsf{T}}^{\beta,0}(\cdot), \tag{3.2}$$

where $\nu_{\pm,\mathsf{T}}^{\beta,B}$ are the Ising measures on tree T with respect to plus (minus) boundary condition. It is further shown there that, when the graphs $\{G_n\}_{n\in\mathbb{N}}$ are *edge-expanders* ,

$$\nu_{n,\pm}^{\beta,0}(\cdot) \to \nu_{\pm,\mathsf{T}}^{\beta,0}(\cdot), \tag{3.3}$$

where $\nu_{n,+}^{\beta,0}(\cdot)$ and $\nu_{n,-}^{\beta,0}(\cdot)$ are the measures (3.1) conditioned to, respectively, $\sum_i x_i{\geq}0$ and $\sum_i x_i{\leq}0$. The latter sharp result provides a better understanding of $\nu_n(\cdot)$, and is much harder to prove than (3.2). For genuinely random limiting trees, in [7] we show that, under a mild but natural continuity assumption, the convergence in (3.2) and (3.3) continue to hold, where now T is chosen according to the limiting tree measure, thereby establishing the claimed universality of this phenomenon.

Most of the proofs of [28] involves explicit calculations which rely on the k-regularity of the graphs G_n and the limiting tree, and thereby does not extend to our set-up. Proof of (3.3) relies on choosing certain functionals of the spin configurations on G_n, which approximate the indicator on the vertices that are in "$-$ state", and whose values concentrate. At the level of generality of our setting the only tools are *unimodularity* of the law of the limiting tree and properties of simple random walk on it. Using these tools we successfully come up with some choices of functionals, based on the *average occupation measure* of the simple random walk on the tree, and prove (3.3) holds under the same continuity property, for *any* edge-expander G_n's. We also show that the continuity assumption holds for all limiting measures for large values of β, if the minimum degree is bounded below by 2. We further confirm that, subject to minimal degree at least 3, the configuration models corresponding to Multitype Galton Watson (MGW) trees are edge-expanders, thereby our theorem applies for most naturally appearing locally tree-like graphs.

After proving the above results for Ising model, one naturally asks whether an analogue of (3.2) and (3.3) hold for the Potts model. The Potts model on a finite graph $G_n = (V_n, E_n)$ is a generalization of the Ising model and is given by the probability measure:

$$\nu_n^{\beta,B,q}(\underline{\sigma}) = \frac{1}{Z_n(\beta,B,q)} \exp\Big\{\beta \sum_{(i,j)\in E_n} \delta_{\sigma_i,\sigma_j} + B\sum_{i\in V_n} \delta_{\sigma_i,1}\Big\}, \tag{3.4}$$

where $\underline{\sigma} = \{\sigma_i : i \in V_n\}$ and $\underline{\sigma} \in \{1, 2, \ldots, q\}^{|V_n|}$, for some integer $q \geq 3$. It is believed that when tree-like graphs G_n converge to a k-regular tree T_k, then for $B = 0$ and any $\beta \geq 0$, the measure $\nu_n^{\beta,B,q}(\cdot)$ converges locally weakly to the symmetric mixture of $\nu_{i,\mathsf{T}_k}^{\beta,0}(\cdot)$, where $\nu_{i,\mathsf{T}_k}^{\beta,0}(\cdot)$ is the Potts measure on T_k with i-boundary condition. This gives an analogue of (3.2). An analogue of (3.3) can be formulated similarly. Analyzing the Potts model is challenging than Ising model because of several reasons. One crucial difference between these two models is that for Ising model when $B > 0$ there is only one Ising Gibbs measure on T_k. Whereas for the Potts model there is a *non-uniqueness* region, in which even for $B > 0$, there are uncountably many Potts Gibbs measures in that region (c.f. [17]). Moreover, all the proofs on Ising model are heavily dependent on FKG inequality, which is also lacking in the Potts model. A way to tackle the latter problem is to move to the random cluster model, proving the relevant results there and then use the Edwards-Sokal coupling to move back to the Potts model. However for infinite trees, the infinite volume random cluster model with general boundary condition is not well understood, making the problem challenging. In an ongoing research project [1], I am pursuing this task of obtaining the weak limit of the Potts model for k-regular locally tree-like graphs, by relating it to the random cluster models.

In an another project with Sumit Mukherjee [10], we consider the Ising model on hypercube $\mathcal{H}_d$. That is, the vertex set of the graph $\mathcal{H}_d$ is $\{0, 1\}^d$, and an edge is drawn between two vertices if their hamming distance is one. Our goal is to find the limiting log-partition of this model, appropriately scaled, as $d \to \infty$. It is believed that the *mean field* prediction would hold for this model. That is, the limiting log-partition function would have the same limit as in the Curie-Weiss model. More specifically, the conjecture is that $\frac{1}{n}\log Z_n(\frac{\beta}{d}, B) \to \phi(\beta, B)$ where $n = 2^d$,

$$\phi(\beta,B) := \sup\{\varphi_{\beta,B}(m), m \in [-1,1]\},\ \varphi_{\beta,B}(m) := Bm + \frac{1}{2}\beta m^2 + H\Big(\frac{1+m}{2}\Big),$$

and $H(x) := -x\log x - (1-x)\log(1-x)$. Note that the hypercube $\mathcal{H}_d$ is not a tree-like graph. Therefore none of the techniques developed in [16, 17, 18] is applicable here. On the other hand if we consider an Ising model on the Erdős-Rényi graph $G(n, p_n)$, then as long as $np_n \to \infty$, it can be shown that the limiting log-partition function obeys the mean field prediction. One crucial fact which helps here is the existence of a spectral gap. The hypercube $\mathcal{H}_d$ lacks this property as well, making the problem more challenging. In [10] we prove the conjecture for $\beta < 1$, and now working on $\beta > 1$ to prove the same.

4. Percolation

During the formulation of a simple stochastic model Broadbent and Hammersley [14] formulated the 'percolation model', around half a century ago. Since then this area has been a very active area of research and lots of work has been done to understand this model on different graphs. Various percolation models are very well understood on $\mathbb{Z}^2$. However there are few percolation models on $\mathbb{Z}^2$ which are yet to be understood. For example, consider the following model: On $\mathbb{Z}^2$ each horizontal edge is right-oriented with probability $p \in (0, 1)$, and otherwise left-oriented. Also each vertical edge is oriented upwards with probability p and otherwise oriented downwards. Let $\theta(p)$ denote the probability that the origin 0 is the endpoint of an infinite self-avoiding path oriented away from 0. In [22] it was conjectured that $\theta(p) > 0$ *iff* $p \neq 1/2$. The intuitive reasoning here is similar to the random walk: When $p \neq 1/2$ there will be a drift in one direction which will enable to create a oriented path escaping to infinity. By coupling with oriented percolation on $\overrightarrow{\mathbb{Z}}^2$ it can be shown that for all $p > \overrightarrow{p}_c$, $\theta(p) > 0$. It is also known that if any small positive density of oriented edges is added at random then the process is *supercritical*. However nothing much is known for this model. Due to lack of the monotonicity property here, one does not have the FKG and BK inequalities for this model. Since these inequalities play crucial roles in almost all of the proof, absence of such inequalities makes the model harder to analyze. Using a multi scale analysis, in a current project with Riddhipratim Basu [3] we are trying to prove the aforementioned conjecture.

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