Qualifying Exam in Basic Analysis, August 2012
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 6 questions

1. State and prove the Cauchy-Schwartz inequality.

2. State some reasonable conditions under which a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) satisfies
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)
\]
everywhere on \( \mathbb{R}^2 \) and prove this equality under the condition you give.

3. For \( a_1, \ldots, a_n > 0 \), let the arithmetic mean be
\[
\frac{1}{n} \sum_{i=1}^{n} a_i
\]
and the geometric mean be
\[
\sqrt[n]{\prod_{i=1}^{n} a_i}.
\]
Is one always larger than the other? State a theorem and prove it, or provide suitable counterexamples.

4. State and prove the contraction mapping principle (also known as the Banach fixed point theorem).

5. Let \( f : [0, 1] \rightarrow \mathbb{R} \). Define what it means for \( f \) to be continuous at a point \( x_0 \in (0, 1) \). Then give an example of a function that is discontinuous everywhere except at \( x_0 = \frac{1}{2} \).

6. Let \( f \) be the solution of
\[
f'(x) = -(x^2 e^{-x^2} + 1) f(x)^3, \quad f(0) = 2.
\]
How many solutions does the equation \( f(x) = \sin(x) \) have on \( \mathbb{R} \)?

Part II: 10 points each. Do all 5 questions.

1. Is
\[
f(x) = \frac{1 + x^2}{x \sqrt{1 - \cos \frac{1}{x}}}
\]
in \( L^1(0, 1) \)?

2. Define, for \( x \in [0, 1] \)
\[
f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n^3} \cos(2\pi nx).
\]
Show that \( f \) is well-defined, i.e. the right-hand side converges pointwise for every \( x \in [0, 1] \). Is \( f \) continuous on \([0, 1]\)? Is \( f \) differentiable on \([0, 1]\)? If so, what is the derivative of \( f \)?

3. Let \( f_n \) be a sequence of continuous functions \( [0, 1] \rightarrow \mathbb{R} \) such that each \( f_n \) satisfies \( f_n(0) = 1 \) and is 1-Lipschitz, i.e. \( |f_n(x) - f_n(y)| \leq |x - y| \) for all \( x, y \in [0, 1] \). Show that \( \{f_n\} \) has a subsequence that converges uniformly on \([0, 1]\). [Hint: Find a subsequence that converges point wise in suitably many points (e.g. a countable dense subset of \([0, 1]\)), and then show that it (or a subsequence thereof) converges uniformly on all of \([0, 1]\)].
4. Let, for $a > 0$,

$$f(x, y) = \begin{cases} \frac{x^5 + y^4}{(x^2 + y^2)^a + x^2 y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$ 

Determine for which values of $a$ this function has each of the following properties:

- $f$ is continuous at $(0, 0)$;
- $f$ is differentiable at $(0, 0)$.

5. Let for $n \geq 1$

$$g_n(x) = \frac{x^3}{n + x^4} \tanh \left( \frac{1}{nx^2 + 2} \right),$$

where of course $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Study the point-wise and uniform convergence of the sequence $\{g_n\}$, i.e. determine the subsets of $\mathbb{R}$ where the sequence $\{g_n\}$ converges point-wise, and the subsets of $\mathbb{R}$ where the convergence is uniform.