All answers and statements should be proved; no partial credit is given to answers without justification.

**Part I: 6 points each, do all 6 questions**

1. State and prove the ratio test for convergence of series. You may assume the comparison test.

2. Let \( f \) be a continuous function from \([0, 1]\) to \(\mathbb{R}\). Prove that \( f \) is uniformly continuous.

3. Let \( f \) be a bounded monotone function on \([0, 1]\). Prove that \( f \) is Riemann integrable on \([0, 1]\).

4. Let \( M \) be a set equipped with 2 metrics \( \rho_1 \) and \( \rho_2 \). Prove \( \sigma := \max\{\rho_1, \rho_2\} \) defines a metric on \( M \). Show that a sequence of points in \((M, \rho_1)\) cannot converge to two distinct points in \( M \).

5. If \( \lim_{n \to \infty} a_n = L \), prove \( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j = L \).

6. Let \( f : [0, 1] \to [0, 1] \) satisfy \(|f'(x)| \leq \frac{\sqrt{2}}{2}\). Prove there exists \( a \in [0, 1] \) with \( f(a) = a \).

**Part II: 10 points each. Do all 6 questions.**

1. Let \( f(x) := \int_{1}^{10} e^{tx} \frac{t}{t+1} \, \text{dt} \). Prove that \( f \) is differentiable with derivative \( f'(x) = \int_{1}^{10} e^{tx} \frac{t^2}{t+1} \, \text{dt} \).

2. Let \( f(x) := \sum_{k=0}^{\infty} \frac{\cos(kx)}{k^2+1} \). Prove that \( f \) is continuous. Compute (justify!) \( \int_{0}^{\pi} f(s) \, \text{ds} \).

3. Consider the system of equations

\[
\begin{align*}
xy^2 + xzu + Ayv^2 &= 2 + A \\
u^3yz + 2xv - u^2v^2 &= 2.
\end{align*}
\]

When \( A = 1 \) show that this equation defines \( u(x, y, z), v(x, y, z) \) near \((x, y, z) = (1, 1, 1)\) and \((u, v) = (1, 1)\). Compute \( \frac{\partial u}{\partial y}(1, 1, 1) \). For which values of \( A \) does this equation not determine \( u \) and \( v \) as \( C^1 \) functions of \((x, y, z)\) near \((1, 1, 1)\)?

4. Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence of positive numbers. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of differentiable functions satisfying

\[
f'_n = \frac{1}{1 + a_n f_n^2}, \quad \text{and} \quad f_n(0) = 1.
\]

Show that there exists a subsequence of \( \{f_n\}_{n=1}^{\infty} \) converging uniformly to a continuous function on \([0, 10]\).

5. Let \( \{x_n\}_{n=1}^{\infty} \) be a Cauchy sequence. Suppose that for all \( \epsilon > 0 \), there is some \( n > \frac{1}{\epsilon} \) such that \( |x_n| < \epsilon \). Prove the sequence converges to 0.

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( C^2 \). Suppose that \( f'(0) = 0 \) and \( f''(0) < 0 \). Prove that there is \( \delta > 0 \) such that \( f(x) < f(0) \) for all \( x \in [-\delta, \delta] \setminus \{0\} \).