Part I: 6 points each, answer all 9 questions

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^2$ function and that $g(x) := e^{f(x)}$.

   (a) Prove that a local max or min of $f$ is also a local max or min of $g$.

   (b) Suppose that $x_0$ is a critical point of $f$. Find a relation between the Hessian of $f$ and the Hessian of $g$ at this critical point.

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ are continuously differentiable functions. Let $h(x) = g(f(x))$. State the general chain rule for the gradient matrix $Dh(x)$.

3. Let $A \subset \mathbb{R}^2$ be the bounded region defined as the set of points $(x, y) \in \mathbb{R}^2$ satisfying $0 \leq x \leq 2$ and $0 \leq y \leq x^2$.

   Let $V \subset \mathbb{R}^3$ be the solid region
   \[ V = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in A, \ 0 \leq z \leq xy \} \, . \]

   Express the volume of $V$ as an integral and evaluate the integral.

4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by
   \[ f(x) = \frac{1}{10} \begin{pmatrix} 2 + x_1 + x_2 + x_1x_2^2 \\ 3 + x_1 - x_2 - x_1^2x_2 \end{pmatrix} \, . \]

   Show that there is a unique $x \in [-1, 1] \times [-1, 1]$ satisfying $f(x) = x$.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $B_r = \{ x : ||x - x_0|| \leq r \}$ be the ball of radius $r$ centered at $x_0$. Show that
   \[ \lim_{r \to 0} \frac{\int_{B_r} f(x) \, dx}{\text{vol} (B_r)} = f(x_0) \, . \]

6. Let $f(x) = x^6 - 3 \sin(\pi x/2) + 2 + \epsilon x$. Prove that if $|\epsilon|$ is small, there is a point $x_\epsilon \in \mathbb{R}$ near $x_0 = 1$ such that $f(x_\epsilon) = 0$.

7. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that
   \[ |f(x) - f(y)| \leq (x - y)^2 \]

   for all $x, y \in \mathbb{R}$. Prove that $f$ is constant.

8. Let $x_0 = (x_0, y_0)$ and $x_1 = (x_1, y_1)$ be two given points on the plane $\mathbb{R}^2$. Compute the integral
   \[ \int_C x \, dy - y \, dx \]

   where $C$ is the segment (straight line) connecting $x_0$ with $x_1$. 
9. Suppose that \( \{a_n\}_{n=1}^\infty \) is a sequence of real numbers and that series

\[
f(x) = \sum_{n=1}^\infty a_n x^n
\]

converges for some \( x > 0 \). Prove that the series converges absolutely for any \( y \in \mathbb{R} \) such that \( |y| < x \).

**Part II: 10 points each, choose 4 out of 5 questions. Only 4 questions will be counted in your score.**

10. Find all values of \( x \in \mathbb{R} \) for which the following series converges:

\[
f(x) = \sum_{n=1}^\infty (-1)^n \frac{(x + 5)^n}{n 3^n 2}
\]

11. Estimate the following integral:

\[
\int_0^{0.1} \frac{s}{(1-s)^3} \, ds.
\]

12. Let \( a, b, c \) be nonzero real numbers. Find the point on the plane \( ax + by + cz = 1 \) that is closest to the origin.

13. Consider the set \( X = C([0, 1]; \mathbb{R}) \) of continuous real-valued functions on \([0, 1]\), and let

\[
\rho(f, g) = \max_{x \in [0, 1]} |f(x) - g(x)|
\]

(i) Prove that \((X, \rho)\) is a metric space.

(ii) Prove that the metric space \((X, \rho)\) is complete.

(iii) Consider the set

\[
D = \{ f \in X \mid |f(x)| \leq 1 \ \forall x \in [0, 1] \}.
\]

Is this set compact in \( X \) (under the metric topology induced by \( \rho \))? Explain.

14. Let \( f(x) : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function with \( f'(0) \neq 0 \) and satisfying

\[
\left| 1 - \frac{f'(x)}{f'(0)} \right| \leq \lambda < 1, \quad \forall x \in [-M, M]
\]

and \( |f(0)| < M(1 - \lambda)|f'(0)| \). Prove that the sequence \( \{x_n\}_{n=1}^\infty \) defined by

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(0)}, \quad n = 0, 1, 2, \ldots
\]

with \( x_0 = 0 \), converges to a point \( x_* \in [-M, M] \) that solves \( f(x_*) = 0 \). Can there be another solution in the interval \([-M, M]\)?