Part I: 6 points each, do all 6 questions

1. State and prove the alternating series test for convergence of series. You may assume the comparison test.

2. Find a solution of the form \( u(t) = \sum_{k=1}^{\infty} u_k \sin(kt), \ \{u_k\}_{k=1}^{\infty} \subset \mathbb{R}, \) to the equation
\[ -u''(t) + u(t) = 1, \quad \text{and} \quad u(0) = u(\pi) = 0. \]

3. Let \( \{q_n\}_{n=1}^{\infty} \) be an enumeration of the rational numbers in \([0, 1].\) Let \( t_n(x) = 0 \) for \( x \in [0, q_n) \) and \( t_n(x) = \frac{1}{n^2} \) for \( x \in [q_n, 1]. \) Set \( h(x) := \sum_{n=1}^{\infty} t_n(x). \) Prove \( h \) is continuous at every irrational number.

4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be \( C^3. \) Prove
\[ \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} \left( f(a + r \cos(\theta), b + r \sin(\theta)) - f(a, b) \right) d\theta = \frac{f_{xx}(a, b) + f_{yy}(a, b)}{4}. \]

5. Let \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) be two bounded sequences of real numbers. Prove or give a counter example: \( \limsup_{n \to \infty} (x_n + y_n) = \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n. \)

6. Let \( \{f_n\}_{n=1}^{\infty} \subset C([0, 1], \mathbb{R}) \) be a monotone increasing sequence of continuous functions which converges pointwise to a continuous function \( g. \) Prove \( f_n \to g \) uniformly.

Part II: 10 points each. Do all 6 questions.

1. Let \( g : \mathbb{Z}^2 \to \mathbb{R}. \) For which \( p \) does the inequality \( |g(k)| \leq c(1 + |k|)^{-p}, \forall k, \) imply \( \sum_{k \in \mathbb{Z}^2} g(k) < \infty? \) Justify.

2. Let \( \phi : \mathbb{R} \to [-1, 1] \) be \( C^1. \) Let \( \{a_k\}_{k=1}^{\infty} \subset \mathbb{R}. \) Set
\[ b(x) := \sum_{k=0}^{\infty} a_k \phi(kx). \]
Suppose that \( b \) is discontinuous at \( x = 1. \) Prove \( \limsup_{k \to \infty} k^2 |a_k| = \infty. \)

3. For each \( p \geq 1, \) define complete normed vector spaces \( (l_p, \|\|_p) \) consisting of sequences \( s = \{s_k\}_{k=1}^{\infty} \) such that \( \|s\|_p := \left( \sum_{k=1}^{\infty} |s_k|^p \right)^{1/p} < \infty. \) One can show (but don’t today) that if \( v \in l_1 \) and \( a \in l_p, \) then the new sequence \( v * a \) defined by \( (v * a)_k = \sum_{j=1}^{\infty} v_j a_{k-j} \) lies in \( l_p \) with \( \|v * a\|_p \leq \|v\|_1 \|a\|_p. \) Suppose that \( \|v\|_1 \leq \frac{1}{2}. \) Let \( f \in l_p. \) Show that there exists \( a \in l_p \) such that
\[ a + v * a = f. \]

4. Let \( f : \mathbb{R}^4 \to \mathbb{R}^2 \) be given by \( f(x, y, z, w) = (2e^x + y^2 - 3, y \cosh(x) - 4x + 2z - w). \) Then \( f(0, 1, 2, 3) = (3, 2). \) Show that there exists a smooth function, \( g(z, w), \) defined near \((2, 3)\) such that \( g(2, 3) = (0, 1) \) and \( f(g(z, w), z, w) = (3, 2). \) Compute \( dg(2,3). \)

5. Let \( X \) denote the space of continuous functions from \([0, 1] \to \mathbb{R}. \) Let \( B_0(R) = \{ f \in X : \max |f(x)| \leq R \}. \) Is \( B_0(R) \) compact for \( R < \infty? \) Justify.

6. Prove that given \( f \in C([0, 1], \mathbb{R}) \) and \( \epsilon > 0, \exists \{a_j\}_{j=0}^{N} \) (\( N \) depending on \( \epsilon, f) \) such that
\[ |\sum_{j=0}^{N} a_j e^{ix} - f(x)| < \epsilon, \forall x \in [0, 1]. \]