My primary objects of study are binary quadratic forms and binary cubic forms. These are polynomials ax2 + bxy + cy2 or ax3 + bx2y + cxy2 + dy3 where a, b, c, and d are integers. Over 200 years ago, Gauss discovered a way of combining two binary quadratic forms to get a third. Extending work of Bhargava on binary cubic forms, I showed that there is an explicit cubic analogue to Gauss's discovery with binary quadratic forms.

An Analogue of Gauss Composition for Binary Cubic Forms

Ben Nativi

Senior Thesis

$$P(X,Y) = p_1'(x_1,y_1)p_2(x_2,y_2) + p_1(x_1,y_1)p_2'(x_2,y_2),$$

where

$$\begin{aligned} p_1(x_1,y_1) &= -x_1^3 + 3x_1^2y_1 + y_1^3, \\ p_2(x_2,y_2) &= -3x_2^3 + 6x_2^2y_2 - 3x_2y_2^2 + y_2^3, \\ P(X,Y) &= -8X^3 + 15X^2Y - 9XY^2 + 2Y^3. \end{aligned}$$

and

$$\begin{aligned} p_1'(x_1,y_1) &= \frac{3}{2}x_1^3 - \frac{3}{2}x_1^2y_1 + 3x_1y_1^2 + \frac{1}{2}y_1^3, \\ p_2'(x_2,y_2) &= \frac{7}{2}x_2^3 - 6x_2^2y_2 + \frac{9}{2}x_2y_2^2 - \frac{1}{2}y_2^3. \\ X &= x_1x_2 + y_1y_2 \\ Y &= x_1y_2 + y_1x_2 + y_1y_2 \end{aligned}$$

Gauss's composition law on binary quadratic forms is specifically well defined on SL2(Z)-equivalence classes of primitive binary quadratic forms. Likewise, the composition law for cubic forms is defined up to SL2(Z)-equivalence of projective forms. I was able to prove this result on cubic forms by studying the rich interplay between classes of forms and the ideal class groups of quadratic integer rings. The composition law on cubic forms is inherited from a correspondence between classes of projective cubic forms and ideal classes of the 3 torsion of certain ideal class groups.

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