EXAM 1

Math 102, Spring 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________________

1. ____________ (/20 points)

2. ____________ (/20 points)

3. ____________ (/20 points)

4. ____________ (/20 points)

5. ____________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

Total ____________ (/100 points)

1. Find the complete set of solutions to the system of equations

$$\begin{array}{rcrcrcr} 1x & + & 2y & - & z & = & 2 \\ 1x & + & 3y & + & 1z & = & 10 \\ 2x & + & 4y & & & = & 10 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 3 & 1 & 10 \\ 2 & 4 & 0 & 10 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 2 & 6 \end{array}\right) \begin{array}{l} (1) \\ (2) - (1) \\ (3) - 2(1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -14 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} (1) - 2(2) \\ (2) \\ (3)/2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} (1) + 5(3) \\ (2) - 2(3) \\ (3) \end{array}$$

$$x = 1$$
$$y = 2$$
$$z = 3$$

There is a single solution, $(x,y,z) = (1,2,3)$

2. In this problem, let

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 9 \\ 8 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 6 \end{pmatrix}$$

Compute the following:

(a) $\vec{v} \cdot \vec{w}$

$$= 1\cdot 4 + 2\cdot 5 + 3\cdot 6$$

$$= \boxed{32}$$

(b) $\|\vec{v}\|$ and $\|\vec{v}\|$

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \boxed{\sqrt{14}}$$

$$\|\vec{w}\| = \sqrt{4^2 + 5^2 + 6^2} = \boxed{\sqrt{77}}$$

(c) $\cos(\theta)$ (the angle between $\vec{v}$ and $\vec{w}$)

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{32}{\sqrt{14}\sqrt{77}} = \boxed{\frac{32}{7\sqrt{22}}}$$

(d) $A\vec{v}$

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 9 \\ 8 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3+8+15 \\ 2+2+27 \\ 8+2+0 \end{pmatrix} = \begin{pmatrix} 26 \\ 31 \\ 10 \end{pmatrix}$$

(e) AB

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 9 \\ 8 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 25 & 24 & 45 \\ 32 & 37 & 59 \\ 17 & 1 & 11 \end{pmatrix}$$

3. The work below shows a series of row operations applied to an augmented matrix, whose coefficient matrix we will call A. The notes to the right of the augmented matrix indicate the row operations that were used in that step. Use information from the computation below to help you answer the questions on the following two pages.

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 1 & 0 & 1 & 1 & 2 & b_3 \\ 2 & 10 & 32 & -1 & 5 & b_4 \\ 3 & 0 & 3 & -2 & 9 & b_5 \end{array}\right)$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 10 & 30 & -1 & 1 & -2b_1 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3)\text{-}1(1) \\ (4)\text{-}2(1) \\ (5)\text{-}3(1) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 5 & 15 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & -1 & 1 & -2b_1 - 2b_2 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) \\ (4)\text{-}2(2) \\ (5) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & -1 & 1 & -2b_1 - 2b_2 + b_4 \\ 0 & 0 & 0 & -2 & 3 & -3b_1 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2)/5 \\ (3) \\ (4) \\ (5) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & -3b_1 - 2b_2 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & 3 & -5b_1 + 2b_3 + b_5 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3) \\ (4)\text{+}1(3) \\ (5)\text{+}2(3) \end{array}$$

$$\left(\begin{array}{ccccc|l} 1 & 0 & 1 & 0 & 0 & 7b_1 + 4b_2 - 2b_3 - 2b_4 \\ 0 & 1 & 3 & 0 & 0 & \frac{1}{5}b_2 \\ 0 & 0 & 0 & 1 & 0 & -b_1 + b_3 \\ 0 & 0 & 0 & 0 & 1 & -3b_1 - 2b_2 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & 0 & 4b_1 + 6b_2 - b_3 - 3b_4 + b_5 \end{array}\right) \begin{array}{l} (1)\text{-}2(4) \\ (2) \\ (3) \\ (4) \\ (5)\text{-}3(4) \end{array}$$

(a) Find the complete solution to the system of equations $A\vec{x} = \vec{c}$, where $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ and $\vec{c} = (1, 0, 5, 1, 4)$.

Let $\vec{b} = \vec{c}$, then $A\vec{x} = \vec{c}$ is same as on prev. page.

$b_1 = 1$, $b_2 = 0$, $b_3 = 5$, $b_4 = 1$, $b_5 = 4$. So we have

$$\begin{aligned} x_1 \quad + x_3 &= -5 \\ x_2 + 3x_3 &= 0 \\ x_4 &= 4 \\ x_5 &= 3 \\ 0 &= 0 \leftarrow \text{consistent!} \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5 - x_3 \\ -3x_3 \\ x_3 \\ 4 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} -5 \\ 0 \\ 0 \\ 4 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$$

(b) Compute the determinant of A.

$\text{rref}(A) \neq I$, so A is <u>not</u> non-singular, thus <u>not</u> invertible. So the determinant must be $\boxed{\text{zero.}}$

(c) Find the precise condition(s) on the components of the vector $\vec{c} = (c_1, \ldots, c_5)$ that will guarantee that the system $A\vec{x} = \vec{c}$ has a solution.

The system will have a solution iff there are no inconsistencies. Given that rref(A) has only one row of zeroes, we only have one constraint.

We need only that

$$4c_1 + 6c_2 - c_3 - 3c_4 + c_5 = 0$$

(d) Are there any cases in which $A\vec{x} = \vec{c}$ has a unique solution? (Make sure to explain your reasoning!)

The variable x_3 is a free variable, so if any solution exists to $A\vec{x} = \vec{c}$ there must be ∞'ly many solutions.

So there can not be any cases where $A\vec{x} = \vec{c}$ has a unique solution.

4. Suppose that a series of row operations reduces the augmented matrix representing the system $A\vec{x} = \vec{b}$ (where $\vec{b} = (b_1, b_2, b_3, b_4)$) to the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1b_1 + 3b_2 + 5b_3 + 1b_4 \\ 0 & 1 & 0 & 0 & 3b_1 + 7b_2 + 2b_3 + 1b_4 \\ 0 & 0 & 1 & 0 & 2b_1 + 1b_2 + 6b_3 + 5b_4 \\ 0 & 0 & 0 & 1 & 5b_1 + 1b_2 + 7b_3 + 8b_4 \end{array}\right)$$

(a) What is the solution to the system $A\vec{x} = (0, 0, 1, 0)$?

Plugging in $\vec{b} = (0,0,1,0)$, we have $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 6 \\ 7 \end{pmatrix}$

(b) The solution to the first part of this problem can be related to one of the columns of A^{-1}, by noting the equivalence

$$A\vec{x} = \vec{e}_3 \quad \Longleftrightarrow \quad \vec{x} = A^{-1}\vec{e}_3$$

and also that for any matrix M, we have

$$M\vec{e}_i = i\text{th column of } M$$

Use this idea to compute the entire matrix A^{-1}.

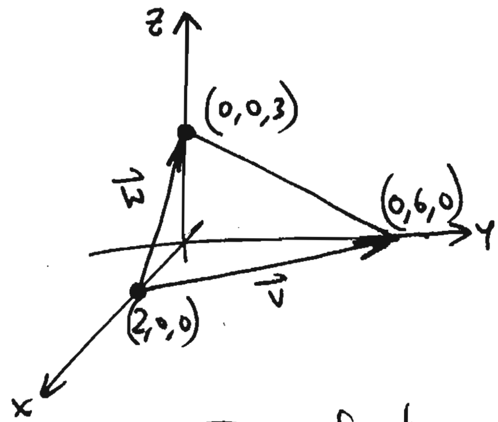
3rd col. of A^{-1} $= A^{-1}\vec{e}_3 =$ sol'n to $A\vec{x} = \vec{e}_3$ $= \begin{pmatrix} 5 \\ 2 \\ 6 \\ 7 \end{pmatrix}$

Similarly for other columns, giving us

$$A^{-1} = \begin{pmatrix} 1 & 3 & 5 & 1 \\ 3 & 7 & 2 & 1 \\ 2 & 1 & 6 & 5 \\ 5 & 1 & 7 & 8 \end{pmatrix}$$

5. The plane P passes through the points $(2,0,0)$, $(0,6,0)$ and $(0,0,3)$ in xyz-space. Let L be the line that is the intersection of P with the plane $y = z$.

(a) Find the point-normal equation for the plane P. (There is more than one possible correct equation.)



$\vec{v} = (-2,6,0)$

$\vec{w} = (-2,0,3)$

Choose $\vec{m} = \vec{v} \times \vec{w} = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -2 & 6 & 0 \\ -2 & 0 & 3 \end{pmatrix}$

$= (18, 6, 12)$

Choose $\vec{n} = \frac{1}{6}\vec{m} = (3,1,2)$

Eqn of plane is $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0$

$$\boxed{3x + y + 2z = 6}$$

(b) The plane P is the graph $z = f(x,y)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Find n, m, and an explicit expression for the function f.

Need $f(x,y)$ so that the eqn of the graph $(z = f(x,y))$ is equivalent to above equation.

$$3x + y + 2z = 6$$

$$z = \frac{6 - 3x - y}{2}$$

So we must have $\boxed{f(x,y) = \frac{6-3x-y}{2}}$

This has two input variables, and one output variable,

so $f : \mathbb{R}^2 \to \mathbb{R}^1$

$$\boxed{n=2, \quad m=1}$$

(c) The plane P is a level set $g = 0$ for a function $g : \mathbb{R}^p \to \mathbb{R}^q$. Find p, q, and an explicit expression for the function g. (There is more than one possible correct function.)

Level sets are in the domain, so we must have

$g : \mathbb{R}^3 \to \mathbb{R}^1$. So $\boxed{p = 3,\ q = 1}$

We need $(g = 0) \Leftrightarrow (3x + y + 2z = 6)$

We can use $\boxed{g(x, y, z) = 3x + y + 2z - 6}$

(d) The line L is represented parametrically by a function $h : \mathbb{R}^r \to \mathbb{R}^s$. Find r, s, and an explicit expression for the function h. (There is more than one possible correct function.)

L is a line in $\mathbb{R}^3$, so it is represented by a function parametrically as

$h : \mathbb{R}^1 \to \mathbb{R}^3$. So $\boxed{r = 1,\ s = 3}$

We need a point in the line, for which we can use any sol'n to
$$3x + y + 2z = 6$$
$$y - z = 0$$
Choose $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

The direction vector $\vec{v}$ is $\parallel$ both planes, so $\perp$ both normal vectors. So, can use $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix}$ or even better the simpler vector $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Then $h(t) = \vec{x}_0 + t\vec{v} = \boxed{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$