EXAM 3

Math 102, Spring 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name ______________________________

ID number ____________________

1. __________ (/20 points)

2. __________ (/10 points)

3. __________ (/30 points)

4. __________ (/20 points)

5. __________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total __________ (/100 points)

1. (a) Determine if the matrix A is positive or negative definite or semidefinite, or indefinite.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix}$$

$$\left.\begin{array}{l} \det(3) = 3 > 0 \\ \det\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 3 > 0 \\ \det\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -5 \end{pmatrix} = -18 < 0 \end{array}\right\} \text{So } A \text{ is } \boxed{\text{indefinite}}$$

(b) Determine if the matrix A above is positive or negative definite or semidefinite, or indefinite, subject to the constraint that $x - 5z = 0$.

$$H = \begin{pmatrix} 0 & 1 & 0 & -5 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -5 & 0 & 1 & -5 \end{pmatrix}$$

$n = 3,\ m = 1$

check last 2 LPM's:

$$LPM_4 = \det H = -(1)\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -5 & 1 & -5 \end{pmatrix} - (-5)\det\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ -5 & 0 & 1 \end{pmatrix}$$

$$= -69$$

$$LPM_3 = \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

Both have same sign as $(-1)^m = -1$, so A is

$\boxed{\text{positive definite on constraint set.}}$

2. Find all of the critical points of the function

$$f(x, y) = x^3 - 3xy^2 + 6y^2$$

$$\nabla f = \begin{pmatrix} 3x^2 - 3y^2 \\ -6xy + 12y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x^2 - 3y^2 = 0 \Rightarrow x = \pm y$$

Case 1: $x = y$

$$-6xy + 12y = 0$$

$$-6y^2 + 12y = 0$$

$$y(-6y + 12) = 0$$

$$y = 0 \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$y = 2 \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Case 2: $x = -y$

$$-6xy + 12y = 0$$

$$6y^2 + 12y = 0$$

$$y(6y + 12) = 0$$

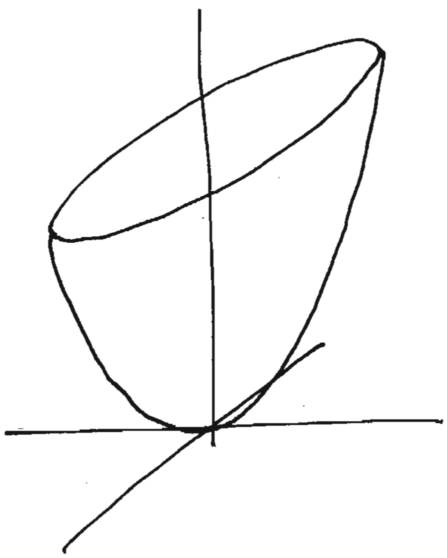
$$y = 0 \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$y = -2 \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Three critical points: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

3. Find the absolute maximum value of the function $f(x, y, z) = x + y + z$ subject to the constraints $z - x - y - 4 \leq 0$ and $x^2 + y^2 - z \leq 0$.

These constraints define the region above a paraboloid and below a plane:



We have four subregions:

Interior (both non-binding)
top face (g_1 binding, g_2 not)
bottom surface (g_1 not, g_2 binding)
edge (both binding)

Interior: Need $Df = 0$. But $Df = \begin{pmatrix}1\\1\\1\end{pmatrix} \neq \vec{0}$, so no critical points there.

Top face: Need $Df = \lambda_1 Dg_1$, $\begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda_1 \begin{pmatrix}-1\\-1\\1\end{pmatrix}$. Not possible, so no crit pts. there.

$\neq \vec{0}$

Bottom surface: Need $Df = \lambda_2 Dg_2$, $\begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda_2 \begin{pmatrix}2x\\2y\\-1\end{pmatrix}$

$\neq \vec{0}$

$\Rightarrow \lambda_2 = -1$, $\Rightarrow x = y = -1/2$

Binding constraint $x^2 + y^2 - z = 0$ then gives $z = 1/2$

So we have one critical point on the bottom surface, at $\begin{pmatrix}-1/2\\-1/2\\1/2\end{pmatrix}$.

(over)

(extra room if needed)

Edge: Need $Df = \lambda_1 Dg_1 + \lambda_2 Dg_2$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda_1 \begin{pmatrix}-1\\-1\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}2x\\2y\\-1\end{pmatrix} \Rightarrow x = y$$

and $z = x+y+4 = 2x+4$

$z = x^2+y^2 = 2x^2$

So $(2x^2) - (2x+4) = 0$

$2(x+1)(x-2) = 0$

$x = -1, 2$

$x = -1 \Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\-1\\2\end{pmatrix}$

$x = 2 \Longrightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\2\\8\end{pmatrix}$

We have three critical points, with values:

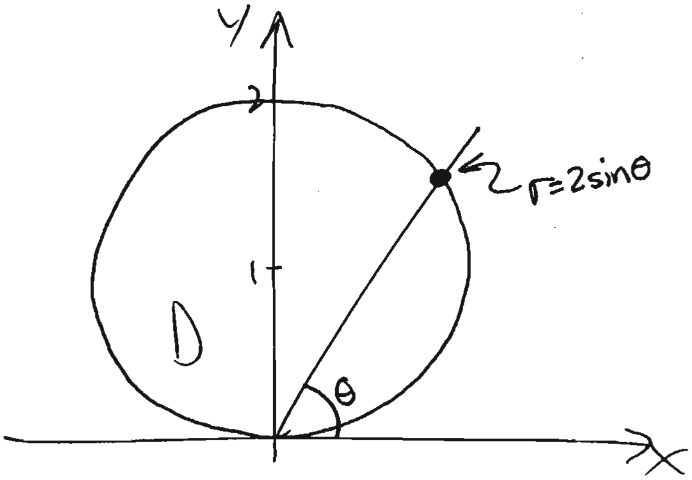
$f\begin{pmatrix}-1/2\\-1/2\\1/2\end{pmatrix} = \frac{1}{2}$

$f\begin{pmatrix}-1\\-1\\2\end{pmatrix} = 0$

$f\begin{pmatrix}2\\2\\8\end{pmatrix} = 12 \leftarrow$ abs. max, attained at $\begin{pmatrix}2\\2\\8\end{pmatrix}$.

4. The price of real estate in a particular rural area is known to be proportional to the distance it is away from a smelly landfill, located at the origin (location, location, location!). Specifically, the price per square kilometer is $P(r) = (\$250,000)(r)$, at a distance r kilometers away from the origin.

A developer would like to purchase a circular plot of land of radius 1km, centered 1km north of the landfill. How much should he expect to have to pay for this purchase?



$$x^2 + (y-1)^2 = 1$$

$$\vdots$$

$$r = 2\sin\theta$$

$$dc = \left(\frac{\text{price}}{\text{area}}\right) dA = P \cdot dA$$

$$C = \iint_D dc = \iint_D P\, dA = \int_0^{\pi}\int_0^{2\sin\theta} P(r)\, r\, dr\, d\theta$$

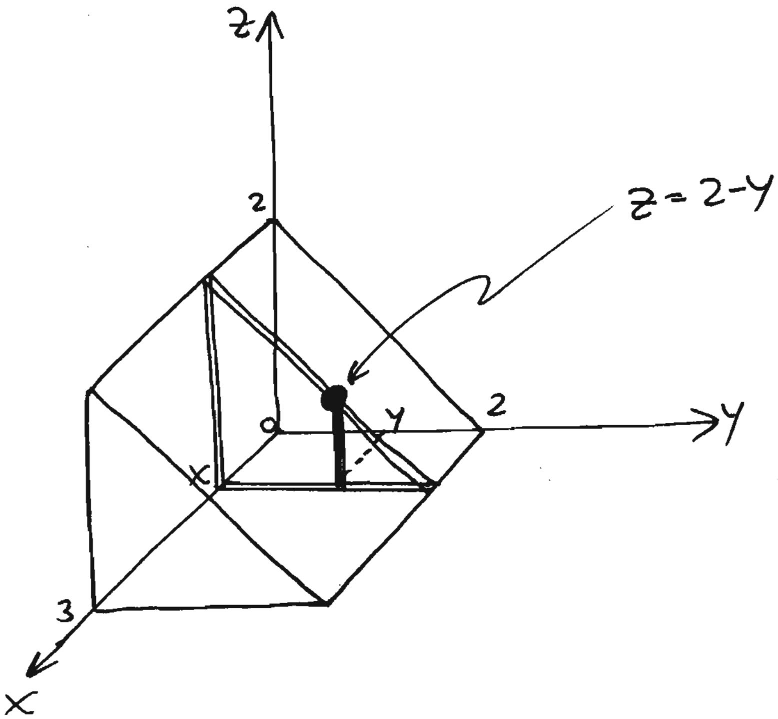
$$= \int_0^{\pi}\int_0^{2\sin\theta} (250{,}000)\, r^2\, dr\, d\theta = \int_0^{\pi} \left(\frac{250{,}000}{3} r^3\right]_0^{2\sin\theta} d\theta$$

$$= \frac{2{,}000{,}000}{3} \int_0^{\pi} \sin^3\theta\, d\theta = \frac{2{,}000{,}000}{3} \int_0^{\pi} \sin\theta - \cos^2\theta \sin\theta\, d\theta$$

$$= \frac{2{,}000{,}000}{3} \left((1-(-1)) - \left(-\frac{1}{3}(\cos\theta)^3\right]_0^{\pi} \right)$$

$$= \frac{2{,}000{,}000}{3} \left(2 - \frac{2}{3} \right) = \boxed{\frac{8{,}000{,}000}{9}}$$

5. Compute the integral of the function $f(x, y, z) = x + yz$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 3$, $z = 2 - y$.



1st $\perp$ x-axis:

$x \in [0,3]$

2nd $\perp$ y-axis:

$y \in [0,2]$

3rd $\perp$ z-axis:

$z \in [0, 2-y]$

$$\iiint f\, dV = \int_0^3 \int_0^2 \int_0^{2-y} (x+yz)\, dz\, dy\, dx$$

$$= \int_0^3 \int_0^2 \left(xz + \frac{1}{2} y z^2 \Big]_{z=0}^{z=(2-y)} \right) dy\, dx$$

$$= \int_0^3 \int_0^2 x(2-y) + \frac{1}{2} y (2-y)^2 \, dy\, dx$$

$$= \int_0^3 \int_0^2 2x - xy + \frac{1}{2} y^3 - 2y^2 + 2y \, dy\, dx$$

$$= \int_0^3 \left(2xy - \frac{1}{2} x y^2 + \frac{1}{8} y^4 - \frac{2}{3} y^3 + y^2 \Big]_{y=0}^{y=2} \right) dx$$

$$= \int_0^3 2x + \frac{2}{3}\, dx = \left(x^2 + \frac{2}{3} x \right]_0^3 = \boxed{\cancel{\ }}\ \boxed{11}$$