# EXAM 1

Math 102, Fall 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name ______________________________

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. Find the complete set of solutions to the system of equations

$$\begin{array}{rcrcrcr} 1x & + & 2y & - & 2z & = & -5 \\ -2x & - & 4y & + & 5z & = & 13 \\ -5x & - & 10y & + & 7z & = & 16 \end{array}$$

2. In this problem, let

$$\vec{v} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix} \; B = \begin{pmatrix} 3 & 4 & -1 \\ -1 & 0 & 2 \\ 1 & 6 & 7 \end{pmatrix}$$

Compute the following:

(a) $\cos(\theta)$ (the angle between $\vec{v}$ and $\vec{w}$)

(b) $A\vec{v}$

(c) $\det(A)$

(d) $\det(ABBA^{-1}BAB^{-1})$

3. (a) Consider the system of equations

$$\begin{array}{rrrrrcl} 1x_1 & +2x_2 & -1x_3 & -1x_4 & +1x_5 & = & b_1 \\ -2x_1 & -4x_2 & +3x_3 & +6x_4 & -6x_5 & = & b_2 \\ -3x_1 & -6x_2 & +0x_3 & -9x_4 & +10x_5 & = & b_3 \\ -1x_1 & -2x_2 & -1x_3 & -7x_4 & +5x_5 & = & b_4 \\ 1x_1 & +2x_2 & -5x_3 & -17x_4 & +13x_5 & = & b_5 \end{array}$$

For which of the following $\vec{b}$ vectors does the above system of equations have at least one solution? Make sure to explain your conclusions.

i. $\vec{b} = (1, 3, 2, 5, 6)$

ii. $\vec{b} = (-2, 1, 17, 4, 2)$

iii. $\vec{b} = (2, -3, -9, -4, -2)$

For your convenience, note that

$$\begin{pmatrix} 30 & 10 & 3 & 0 & 0 \\ 38 & 13 & 4 & 0 & 0 \\ 9 & 3 & 1 & 0 & 0 \\ 23 & 8 & 2 & 1 & 0 \\ 43 & 16 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & -1 & 1 \\ -2 & -4 & 3 & 6 & -6 \\ -3 & -6 & 0 & -9 & 10 \\ -1 & -2 & -1 & -7 & 5 \\ 1 & 2 & -5 & -17 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Suppose that a system of equations $A\vec{x} = \vec{b}$, with ten equations and ten variables, is known not to have any solutions when $\vec{b} = \vec{b_1}$, but does have a solutions when $\vec{b} = \vec{b_2}$. Use pivots to explain how you know that $A\vec{x} = \vec{b_2}$ must have infinitely many solutions.

4. (a) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -7 \\ 2 \\ 0 \\ -4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -1 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Write down a matrix equation whose solutions allow one to determine whether these vectors are linearly independent. Explain what specifically about the solutions allows one to make this determination. Explain why this works.

(b) Find the equation of the unique plane in $\mathbb{R}^3$ that is parallel to, and equidistant from, the two lines parametrized by

$$\begin{bmatrix} 3-2t \\ 2+3t \\ 4-t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5-t \\ 3+t \\ 4+2t \end{bmatrix}$$

5. In this problem we consider the paraboloid $P$ in $xyz$-space with equation given by

$$x^2 + y^2 - z = 5$$

(a) The surface $P$ is the graph $z = f(x, y)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Find $n$, $m$, and an explicit expression for the function $f$.

(b) The surface $P$ is a level set $g^{-1}(0)$ for a function $g : \mathbb{R}^p \to \mathbb{R}^q$. Find $p$, $q$, and an explicit expression for the function $g$. (There is more than one possible correct function.)

(c) The intersection of the surface $P$ with the plane $y = 3$ is parametrized by a function $h : \mathbb{R}^r \to \mathbb{R}^s$. Find $r$, $s$, and an explicit expression for the function $h$. (There is more than one possible correct function.)