EXAM 1

Math 102, Fall 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total __________ (/100 points)

1. Find the complete set of solutions to the system of equations

$$\begin{array}{rcrcrcr} 1x & + & 2y & - & 2z & = & -5 \\ -2x & - & 4y & + & 5z & = & 13 \\ -5x & - & 10y & + & 7z & = & 16 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & -5 \\ -2 & -4 & 5 & 13 \\ -5 & -10 & 7 & 16 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & -9 \end{array}\right) \begin{array}{l} 1 \\ 2 + 2\,1 \\ 3 + 5\,1 \end{array}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} 1 + 2\,2 \\ 2 \\ 3 + 3\,2 \end{array}$$

$z = 3$

$x = 1 - 2y$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2y \\ y \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

2. In this problem, let

$$\vec{v} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \quad A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix} \; B = \begin{pmatrix} 3 & 4 & -1 \\ -1 & 0 & 2 \\ 1 & 6 & 7 \end{pmatrix}.$$

Compute the following:

(a) $\cos(\theta)$ (the angle between $\vec{v}$ and $\vec{w}$)

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{5\cdot 2 + 2\cdot 3 + 1\cdot 0}{\sqrt{5^2+2^2+1^2}\sqrt{2^2+3^2+0^2}} = \frac{16}{\sqrt{30}\sqrt{13}}$$

(b) $A\vec{v}$

$$\begin{pmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 7 \\ 16 \end{pmatrix}$$

(c) $\det(A)$

$$= 2\det\begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix} - 3\det\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} + 0$$

$$= 2\cdot(-1) - 3(-11) = 31$$

(d) $\det(ABBA^{-1}BAB^{-1})$

$$\det B = -(-1)\det\begin{pmatrix} 4 & -1 \\ 6 & 7 \end{pmatrix} + 0 - (2)\det\begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$$

$$= 34 - 28 = 6$$

$$\det(ABBA^{-1}BAB^{-1}) = \det A \det B \det B \det A^{-1} \det B \det A \det B^{-1}$$

$$= \cancel{31}\cdot 6 \cdot 6 \cdot \cancel{\tfrac{1}{31}} \cdot \cancel{6} \cdot 31 \cdot \cancel{\tfrac{1}{6}}$$

$$= 36\cdot 31 = \boxed{1116}$$

3. (a) Consider the system of equations

$$\begin{array}{rrrrrcl} 1x_1 & +2x_2 & -1x_3 & -1x_4 & +1x_5 & = & b_1 \\ -2x_1 & -4x_2 & +3x_3 & +6x_4 & -6x_5 & = & b_2 \\ -3x_1 & -6x_2 & +0x_3 & -9x_4 & +10x_5 & = & b_3 \\ -1x_1 & -2x_2 & -1x_3 & -7x_4 & +5x_5 & = & b_4 \\ 1x_1 & +2x_2 & -5x_3 & -17x_4 & +13x_5 & = & b_5 \end{array}$$

For which of the following $\vec{b}$ vectors does the above system of equations have at least one solution? Make sure to explain your conclusions.

i. $\vec{b} = (1, 3, 2, 5, 6)$
ii. $\vec{b} = (-2, 1, 17, 4, 2)$
iii. $\vec{b} = (2, -3, -9, -4, -2)$

For your convenience, note that

given: this is the rref of

$$\underbrace{\begin{pmatrix} 30 & 10 & 3 & 0 & 0 \\ 38 & 13 & 4 & 0 & 0 \\ 9 & 3 & 1 & 0 & 0 \\ 23 & 8 & 2 & 1 & 0 \\ 43 & 16 & 4 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 2 & -1 & -1 & 1 \\ -2 & -4 & 3 & 6 & -6 \\ -3 & -6 & 0 & -9 & 10 \\ -1 & -2 & -1 & -7 & 5 \\ 1 & 2 & -5 & -17 & 13 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{R=\text{rref}(A)}$$

A is the above coefficient matrix, and R is the r.r.e.f., so E is the product of elementary matrices used in the row reduction. So the system $A\vec{x}=\vec{b}$ above reduces from

$$A\vec{x} = \vec{b}$$

to

$$EA\vec{x} = E\vec{b}$$

or

$$R\vec{x} = E\vec{b}$$

R has two rows of zeroes, so we need the last two components of $E\vec{b}$ to be zeroes to have solutions. So we have:

i) $\vec{b} = (1,3,2,5,6) \Rightarrow E\vec{b} = (\cdot,\cdot,\cdot,56,105) \Rightarrow$ no sol'ns

ii) $\vec{b} = (-2,1,17,4,2) \Rightarrow E\vec{b} = (\cdot,\cdot,\cdot,0,0) \Rightarrow$ solutions exist

iii) $\vec{b} = (2,-3,-9,-4,-2) \Rightarrow E\vec{b} = (\cdot,\cdot,\cdot,0,0) \Rightarrow$ solutions exist

(b) Suppose that a system of equations $A\vec{x} = \vec{b}$, with ten equations and ten variables, is known not to have any solutions when $\vec{b} = \vec{b_1}$, but does have a solutions when $\vec{b} = \vec{b_2}$. Use pivots to explain how you know that $A\vec{x} = \vec{b_2}$ must have infinitely many solutions.

If $A\vec{x} = \vec{b}_1$ has no solutions, there must be a contradiction in the r.r.e.f., which means rref (A) has at least one row of zeroes. So there are at most 9 pivots, and thus at least one column with no pivot.

So there is a free variable.

Thus, since $A\vec{x} = \vec{b}_2$ has a solution, there must be infinitely many solutions.

4. (a) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -7 \\ 2 \\ 0 \\ -4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ -1 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Write down a matrix equation whose solutions allow one to determine whether these vectors are linearly independent. Explain what specifically about the solutions allows one to make this determination. Explain why this works.

The system is

$$\begin{pmatrix} 1 & -7 & 2 & 4 \\ 3 & 2 & 1 & 1 \\ 6 & 0 & 5 & 2 \\ 2 & 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The vectors are l.d. iff this system has multiple solutions.

Reason: Because $\vec{c} = \vec{0}$ is clearly a solution, we have that there are multiple solutions iff there is a nonzero solution. Which,

$\iff$ there is a nonzero l.c.

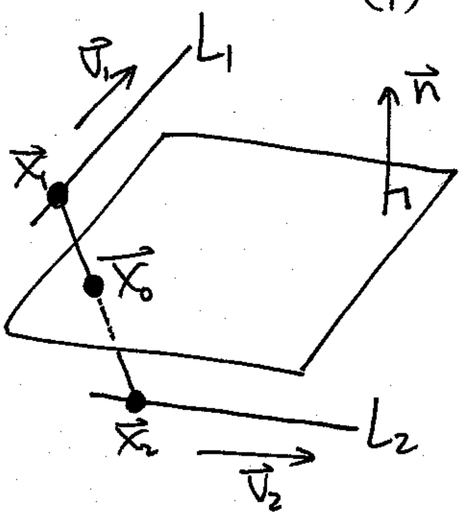
$$c_1\vec{v}_1 + \dots + c_4\vec{v}_4 = \vec{0}$$

$\iff \{\vec{v}_1, \dots, v_4\}$ is l.d.

(b) Find the equation of the unique plane in $\mathbb{R}^3$ that is parallel to, and equidistant from, the two lines parametrized by

$$\begin{bmatrix} 3-2t \\ 2+3t \\ 4-t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5-t \\ 3+t \\ 4+2t \end{bmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$



$\vec{x}_0$ is the midpt btw $\vec{x}_1, \vec{x}_2$,

so $$\vec{x}_0 = \frac{\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}}{2} = \begin{pmatrix} 4 \\ 5/2 \\ 4 \end{pmatrix}$$

$\vec{n}$ is $\perp$ to both $\vec{v}_1, \vec{v}_2$, so choose

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -2 & 3 & -1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix}$$

Equation is then:

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$\begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5/2 \\ 4 \end{pmatrix}$$

$$\boxed{7x + 5y + z = \frac{89}{2}}$$

5. In this problem we consider the paraboloid P in xyz-space with equation given by

$$x^2 + y^2 - z = 5$$

(a) The surface P is the graph $z = f(x, y)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Find n, m, and an explicit expression for the function f.

$$\underset{(x,y)}{f : \mathbb{R}^2} \to \underset{(z)}{\mathbb{R}^1}, \text{ so } \boxed{n=2,\ m=1}$$

Solving for z gives $z = x^2+y^2-5$

So $\boxed{f(x,y) = x^2+y^2-5}$

(b) The surface P is a level set $g^{-1}(0)$ for a function $g : \mathbb{R}^p \to \mathbb{R}^q$. Find p, q, and an explicit expression for the function g. (There is more than one possible correct function.)

$$x^2+y^2-z=5 \iff \underbrace{x^2+y^2-z-5}_{g} = 0$$

So $\boxed{g(x,y,z) = x^2+y^2-z-5}$

$g : \mathbb{R}^3 \to \mathbb{R}^1$ so $\boxed{p=3,\ q=1}$

(c) The intersection of the surface P with the plane $y = 3$ is parametrized by a function $h : \mathbb{R}^r \to \mathbb{R}^s$. Find r, s, and an explicit expression for the function h. (There is more than one possible correct function.)

Intersection is defined by two equations:

$$x^2 + y^2 - z = 5$$

and $y = 3$

So we have $y = 3$, and so

$$x^2 + 3^2 - z = 5$$

$$z = x^2 + 4$$

Then we can write all three coordinates in terms of x:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 3 \\ x^2+4 \end{pmatrix}$$

Letting $x = t$, we have

$$\vec{x}(t) = \boxed{\begin{pmatrix} t \\ 3 \\ t^2+4 \end{pmatrix} = h(t)}$$

$$\underset{(t)}{h : \mathbb{R}^1} \to \underset{(x,y,z)}{\mathbb{R}^3}$$ so $\boxed{r = 1,\ s = 3}$