EXAM 2

Math 102, Fall 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number ____________

1. ________ (/20 points)

2. ________ (/20 points)

3. ________ (/20 points)

4. ________ (/20 points)

5. ________ (/20 points)

Total ________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

1. (a) Let $f(w,x,y,z) = (\tan(wx - yz), wxe^{yz})$. Use the total derivative of f to estimate the value $f(.99, 2.03, 1.98, 1.01)$.

$$(w^*, x^*, y^*, z^*) = (1,2,2,1)$$

$$(dw, dx, dy, dz) = (-.01, .03, -.02, .01)$$

$$Df = \begin{pmatrix} x\sec^2(wx-yz) & w\sec^2(wx-yz) & -z\sec^2(wx-yz) & -y\sec^2(wx-yz) \\ xe^{yz} & we^{yz} & wxze^{yz} & wxye^{yz} \end{pmatrix}$$

$$(Df)(w^*,x^*,y^*,z^*) = \begin{pmatrix} 2 & 1 & -1 & -2 \\ 2e^2 & e^2 & 2e^2 & 4e^2 \end{pmatrix}$$

$$f(w,x,y,z) \approx f(w^*,x^*,y^*,z^*) + \left((Df)(w^*,x^*,y^*,z^*)\right)(dw,dx,dy,dz)$$

$$\approx \begin{pmatrix} 0 \\ 2e^2 \end{pmatrix} + \begin{pmatrix} .01 \\ .01e^2 \end{pmatrix} \approx \boxed{\begin{pmatrix} .01 \\ 2.01e^2 \end{pmatrix}}$$

(b) Find the equation of the tangent plane to the surface described by the equation

$$x^2e^y + xz^2 = 6$$

at the point $(2, 0, 1)$.

Surface is a level set of $g(x,y,z) = x^2e^y + xz^2$

$$\nabla g = \begin{pmatrix} 2xe^y + z^2 \\ x^2e^y \\ 2xz \end{pmatrix} \qquad \nabla g(2,0,1) = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$$

$\nabla g \perp$ level set, so we can use this as our normal vector $\vec{n}$.

Equation of tangent plane then is

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0$$

$$\boxed{5x + 4y + 4z = 14}$$

2. Suppose we have $x = e^{u^2v^2-v}$, $y = e^{u^2-v^2}$, $p = e^{x^4-y^2}$, $q = e^{x^2-y^3}$.

(a) Compute the derivative matrix for the function $g(u, v) = (x, y)$.

$$g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{u^2v^2-v} \\ e^{u^2-v^2} \end{pmatrix}$$

$$D_g = \begin{pmatrix} (2uv^2)e^{u^2v^2-v} & (2u^2v-1)e^{u^2v^2-v} \\ (2u)e^{u^2v^2-v} & (-2v)e^{u^2-v^2} \end{pmatrix}$$

(b) Compute the derivative matrix for the function $f(u, v) = (p, q)$ when $(u, v) = (1, 1)$.

Let $h(x,y) = (p,q)$, so that $f = h \circ g$

We compute Dh as

$$Dh = \begin{pmatrix} 4x^3 e^{x^4-y^2} & -2ye^{x^4-y^2} \\ 2x\, e^{x^2-y^3} & -3y^2e^{x^2-y^3} \end{pmatrix}$$

When $(u,v) = (1,1)$, we have $(x,y) = (1,1)$, and so

$$Dh\Big|_{(1,1)} = \begin{pmatrix} 4 & -2 \\ 2 & -3 \end{pmatrix} \text{ and } Dg\Big|_{(1,1)} = \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}$$

By the Chain Rule:

$$Df(1,1) = Dh(1,1)\, D_g(1,1)$$

$$= \begin{pmatrix} 4 & 8 \\ -2 & 8 \end{pmatrix}$$

3. (a) Use the gradient vector to compute the directional derivative of the function $f(x, y) = x^2 + xy$ at the point $(1, 2)$, in the unit vector direction of the vector $(5, 12)$.

$$\nabla f = \begin{pmatrix} 2x+y \\ x \end{pmatrix} \qquad \vec{v} = (5, 12)$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{5}{13}, \frac{12}{13}\right)$$

$$\nabla f(1,2) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5/13 \\ 12/13 \end{pmatrix} = \boxed{\frac{32}{13}}$$

(b) Compute the above directional derivative directly from the definition of the directional derivative.

$$D_{\vec{u}} f(1,2) = \frac{d}{dt}\Big|_{t=0} f\left((1,2) + t\left(\frac{5}{13}, \frac{12}{13}\right)\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 1 + \frac{5t}{13} \\ 2 + \frac{12t}{13} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \left(1 + \frac{5t}{13}\right)^2 + \left(1 + \frac{5t}{13}\right)\left(2 + \frac{12t}{13}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} 1 + \frac{10t}{13} + \frac{25t^2}{169} + 2 + \frac{10t}{13} + \frac{12t}{13} + \frac{60t^2}{169}$$

$$= \frac{d}{dt}\Big|_{t=0} \left(3 + \left(\frac{10}{13} + \frac{10}{13} + \frac{12}{13}\right)t + \left(\frac{25}{169} + \frac{60}{169}\right)t^2\right)$$

$$= \left(\left(\frac{32}{13}\right) + 2\left(\frac{85}{169}\right)t\right)\Big|_{t=0} = \boxed{\frac{32}{13}}$$

4. Consider the system of equations

$$\begin{aligned} F_1 = \quad v^2x + wxy - xz &= -1 \\ F_2 = \quad vw + vx + wx + xz &= 1 \end{aligned}$$

near the point $(v^*, w^*, x^*, y^*, z^*) = (0, 0, 1, 1, 1)$.

(a) Show that we can locally view the variables v and w as functions of the variables x, y, and z.

$$D_{vw}\vec{F}\Big|_{(0,0,1,1,1)} = \begin{pmatrix} 2vx & xy \\ w+x & v+x \end{pmatrix}\Bigg|_{(0,0,1,1,1)} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\det\left(D_{vw}\vec{F}\right) = -1 \neq 0$$

So, by the implicit function theorem, we can view v, w locally as fns of x, y, z.

(b) Viewing the variables as above, compute $\frac{\partial w}{\partial x}$.

Differentiating the above system w.r.t. x, we get

$$\left(D_{vw}\vec{F}\right)\begin{pmatrix} \partial v/\partial x \\ \partial w/\partial x \end{pmatrix} + \begin{pmatrix} \partial F_1/\partial x \\ \partial F_2/\partial x \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \partial v/\partial x \\ \partial w/\partial x \end{pmatrix} + \begin{pmatrix} v^2+wy-z \\ v+w+z \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \partial v/\partial x \\ \partial w/\partial x \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{0}$$

By Cramer's rule,

$$\frac{\partial w}{\partial x} = \frac{\det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}{\det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} = \frac{-1}{-1} = \boxed{1}$$

5. (a) Given the matrix A below, determine if the quadratic form $Q(\vec{x}) = \vec{x} \cdot A\vec{x}$ is positive or negative definite or semidefinite, or indefinite.

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

The leading principal minors are:

$M_1 = 1 \qquad M_{12} = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0 \qquad M_{123} = \det A = 0$

So Q must be semidefinite; continuing to check all principal minors:

$M_2 = 1 \qquad M_3 = 4$

$M_{13} = \det\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = 0 \qquad M_{23} = \det\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = 0$

All principal minors are ≥ 0, so Q is $\boxed{\text{positive semidefinite}}$

(b) Find all of the critical points of the function $f(x, y) = x^2 - xy^2 + 3y + y^4$.

$$\nabla f = \begin{pmatrix} 2x - y^2 \\ -2xy + 3 + 4y^3 \end{pmatrix} = \vec{0}$$

We first note that $2x - y^2 = 0$, so we must have $2x = y^2$. Plugging this into the bottom, we get

$$-2xy + 3 + 4y^3 = 0$$

$$-y^3 + 3 + 4y^3 = 0$$

$$3y^3 = -3$$

$$y = -1$$

$$x = \frac{y^2}{2} = \frac{1}{2}$$

So the only critical point is

$$(x, y) = \boxed{\left(\frac{1}{2}, -1\right)}$$