EXAM 3

Math 102, Fall 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________
2. ________
3. ________
4. ________
5. ________

Total Score ________ (/100 points)

1. *(20 pts)* Find the shortest distance between the origin and a point on the surface defined by $z = e^{-(x^2+y^2)}$.

Objective: $f = x^2+y^2+z^2$ Constraint: $h = z - e^{-(x^2+y^2)} = 0$

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \qquad \nabla h = \begin{pmatrix} 2x\,e^{-(x^2+y^2)} \\ 2y\,e^{-(x^2+y^2)} \\ 1 \end{pmatrix}$$

Degeneracy: Need $\nabla h = \vec{0}$; not possible.

Lagrange Condition: $\nabla f = \lambda \nabla h$

$$\Rightarrow \left.\begin{aligned} 2x &= \lambda \cdot 2x e^{-(x^2+y^2)} \\ 2y &= \lambda \cdot 2y e^{-(x^2+y^2)} \end{aligned}\right\} \Rightarrow \lambda = e^{(x^2+y^2)} \quad \text{or} \quad x = y = 0$$

$$2z = \lambda \cdot 1$$

and $z = e^{-(x^2+y^2)}$

$\lambda = e^{(x^2+y^2)} \Rightarrow \lambda = 1/z \Rightarrow 2z = 1/z \Rightarrow z = \sqrt{1/2}$

$x = y = 0 \Rightarrow z = 1 \Rightarrow$ crit. pt.: $(0,0,1)$

If $z = \sqrt{1/2}$, then $h = 0 \Rightarrow (x^2+y^2) = \frac{1}{2}\ln 2$

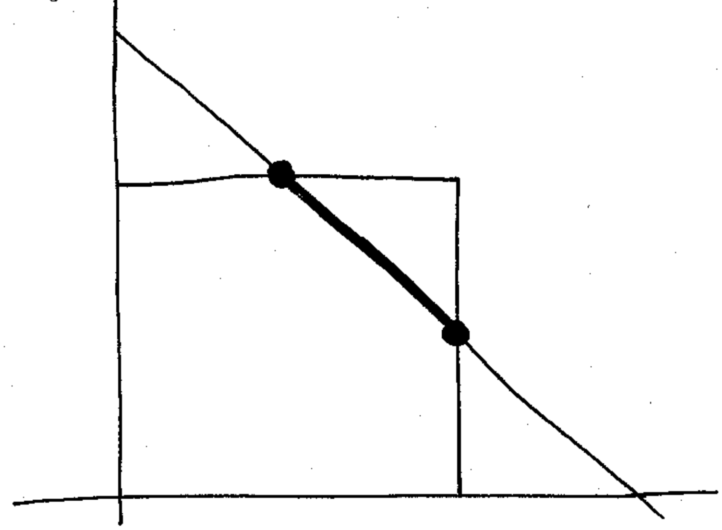
$\Rightarrow x^2+y^2+z^2 = \frac{1}{2}(1+\ln 2)$

$\Rightarrow \text{dist} = \sqrt{\frac{1}{2}(1+\ln 2)}$

At other crit. pt., $d = 1 > \sqrt{\frac{1}{2}(1+\ln 2)}$

So, shortest distance is $\boxed{\sqrt{\frac{1}{2}(1+\ln 2)}}$

2. *(20 pts)* Find the absolute maximum value of the function $f(x, y) = xy$ subject to the constraints $x \geq 0$, $y \geq 0$, $x \leq 2$, $y \leq 2$, $x + y = 3$.



Domain is just the pictured line segment.

Both endpoints are critical.

On segment:

1 constraint, $x+y=3$, let $h(x,y) = x+y-3 = 0$

$$\nabla f = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \nabla h = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ never } \vec{0}!$$

$$\nabla f = \lambda \nabla h \Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x = y \quad \Rightarrow x = y = \frac{3}{2}$$

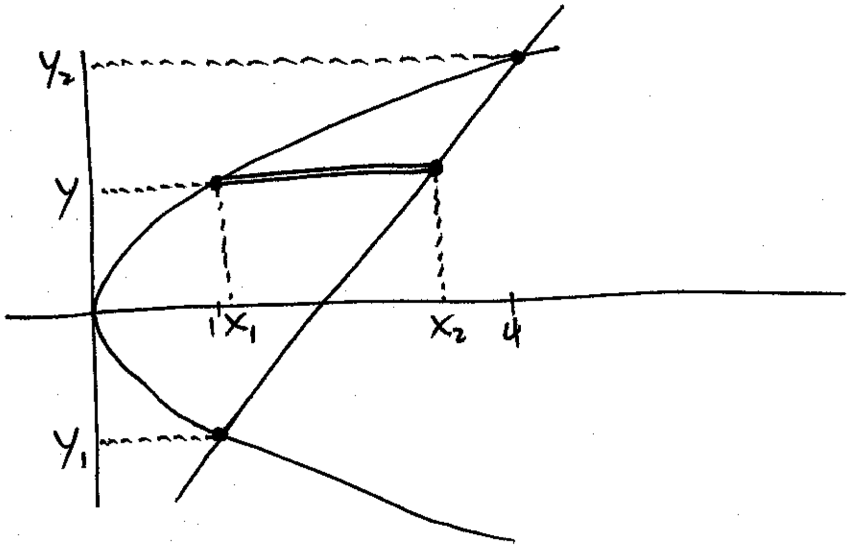
Critical points are:

$(1, 2)$, $f = 2$

$(2, 1)$, $f = 2$

$\left(\frac{3}{2}, \frac{3}{2}\right)$, $f = \boxed{\frac{9}{4}}$ absolute maximum

3. *(20 pts)* Compute the integral of the function $f(x, y) = x - y^2$ over the domain that is bounded by the curves $x = y^2$ and $y = x - 2$.



y_1, y_2 on both $x = y^2$ and $y = x - 2$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow y_1 = -1, y_2 = 2$

(x_1, y) on $x = y^2$, so $x_1 = y^2$

(x_2, y) on $y = x - 2$, so $x_2 = y + 2$

$$\iint_D x - y^2 \, dA = \int_{-1}^{2}\int_{y^2}^{y+2} x - y^2 \, dx\, dy$$

$$= \int_{-1}^{2} \left(\frac{1}{2}x^2 - xy^2 \Big]_{x=y^2}^{x=y+2} \right) dy$$

$$= \int_{-1}^{2} \left(\frac{1}{2}(y+2)^2 - (y^3 + 2y^2)\right) - \left(\frac{1}{2}y^4 - y^4\right) dy$$

$$= \int_{-1}^{2} \frac{1}{2}y^4 - y^3 - \frac{3}{2}y^2 + 2y + 2 \, dy$$

$$= \frac{1}{10}y^5 - \frac{1}{4}y^4 - \frac{1}{2}y^3 + y^2 + 2y \Big]_{-1}^{2} = \left(\frac{32}{10} - 4 - 4 + 4 + 4\right) - \left(\frac{-1}{10} - \frac{1}{4} + \frac{1}{2} + 1 - 2\right)$$

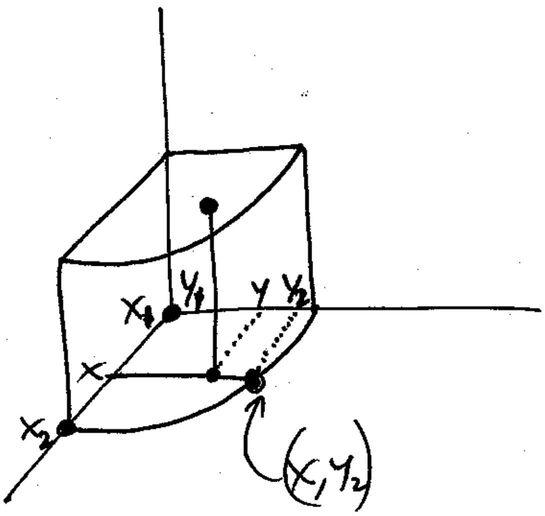
$$= \boxed{81/20}$$

4. *(20 pts)* The "probability density function" for the governmentally determined economic parameters x, y, and z is given by $f(x,y,z) = 4xy$. A stock speculator believes that the price of a particular stock will go up if those parameters fall in the domain D defined by $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$, and $0 \leq z \leq 1$.

The probability of this happening is given by the integral

$$\iiint_D f(x,y,z)\,dV$$

Compute this probability.



on $x^2+y^2=1$

$\Rightarrow y_2 = \sqrt{1-x^2}$

Slice $\perp$ x-axis:

$x \in [x_1, x_2] = [0,1]$

Slice $\perp$ y-axis:

$y \in [y_1, y_2] = [0, \sqrt{1-x^2}]$

Slice $\perp$ z-axis:

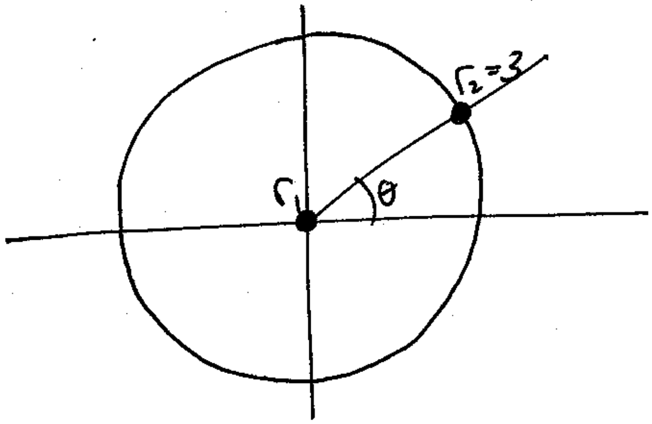
$z \in [z_1, z_2] = [0,1]$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 4xy \quad dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \Big(4xyz\Big]_{z=0}^{z=1} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} 4xy\,dy\,dx = \int_0^1 \Big(2xy^2\Big]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 2x - 2x^3\,dx = x^2 - \frac{1}{2}x^4\Big]_0^1 = \boxed{1/2}$$

5. *(20 pts)* The population of a circular island (of radius 3 miles) is concentrated at the center of the island, with the population density (in thousands of people per square mile) given by $\delta(x, y) = 4e^{-r^2}$, where $r^2 = x^2 + y^2$. Find the total population of the island.



$\theta \in [0, 2\pi]$

$r \in [r_1, r_2] = [0, 3]$

$$P = \iint_D dP = \iint_D \delta(x,y)\, dA = \iint 4e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 4e^{-r^2} r\; dr\, d\theta$$

let $u = -r^2$
$du = -2r\,dr$

$$= \int_0^{2\pi}\left(\int (-2e^u)\,du\right) d\theta = \int_0^{2\pi} (-2e^u)\Big]\; d\theta$$

$$= \int_0^{2\pi} \left(-2e^{-r^2}\right]_0^3 d\theta = \int_0^{2\pi} 2(1-e^{-9})\; d\theta$$

$$= 2(1-e^{9})\,\theta\Big]_0^{2\pi} = \boxed{4\pi(1-e^{-9})}$$