EXAM 1

Math 102, Spring 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________
8. __________
9. __________
10. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(12 pts)* Find the complete set of solutions to the system of equations

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & 2z & = & 12 \\ 3x & + & 7y & + & 5z & = & 23 \\ 2x & + & 2y & - & 2z & = & 2 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & 12 \\ 3 & 7 & 5 & 23 \\ 2 & 2 & -2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 3 & 7 & 5 & 23 \\ 2 & 2 & -2 & 2 \end{array}\right) \begin{array}{l} ①/2 \\ ② \\ ③ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & -4 & -10 \end{array}\right) \begin{array}{l} ① \\ ② - 3① \\ ③ - 2① \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ① - 2② \\ ② \\ ③ + 2② \end{array}$$

RREF x, y pivot vars
z free

$x = -4 + 3z$
$y = 5 + 2z$
$z = z$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 + 3z \\ 5 + 2z \\ z \end{pmatrix}$$

For the problems on this page, let

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad A = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 5 & 6 \\ 2 & 0 & 3 \end{pmatrix} B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ -2 & 5 & 7 \end{pmatrix}$$

Find the following:

2. *(8 pts)* $\cos(\theta)$ (where θ is the angle between $\vec{v}$ and $\vec{w}$)

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \frac{3 \cdot 1 + 1 \cdot 1 + (-3) \cdot 2}{\sqrt{19}\,\sqrt{6}} = \frac{-2}{\sqrt{114}}$$

3. *(8 pts)* AB

$$= \left(A_i \cdot b_j \right)_{ij} = \begin{pmatrix} 6 & 10 & 10 \\ -3 & 32 & 49 \\ -2 & 17 & 23 \end{pmatrix}$$

4. *(8 pts)* $\det(A)$ $= 2 \det\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix} - 0 \det\begin{pmatrix} 5 & 1 \\ 2 & 6 \end{pmatrix} + 3 \det\begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix}$

$$= (2)(-17) + 0 + (3)(29)$$

$$= 53$$

5. *(8 pts)* A parametric representation of the line passing through the points $\vec{v}$ and $\vec{w}$.

$$\vec{X} = \vec{X_0} + t\vec{u}$$

$$= \vec{v} + t(\vec{w} - \vec{v}) = \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$$

6. *(12 pts)* Suppose we know that the unique solution to the system

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is $(x, y, z) = (2, 1, 0)$.

Can you ensure that the system below has at least one solution? Explain why or why not.

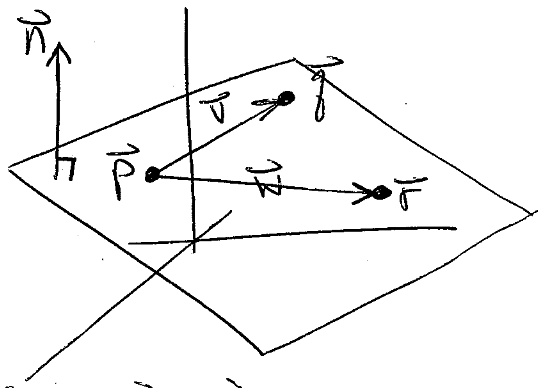
$$\underbrace{\begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}}_{B_1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 9 \end{bmatrix}$$

Because the given system has unique solutions, we know $\det A \neq 0$. Cramer's rule then gives us

$$x = 2 = \frac{\det B_1}{\det A}$$

From this we know $\det B_1 \neq 0$, so B_1 is non-singular. ~~...~~ This means that the system $B_1\vec{x} = \vec{c}$ has both existence and uniqueness.

7. *(10 pts)* Let P be the unique plane that passes through the points $\vec{p} = (2,1,5)$, $\vec{q} = (1,3,5)$, $\vec{r} = (2,2,6)$. Find the standard equation of the plane P.



$$\vec{v} = \vec{q} - \vec{p} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\vec{w} = \vec{r} - \vec{p} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{n} = \vec{v} \times \vec{w} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Use $\vec{X_0} = \vec{p}$.

Eq. of plane is:

$$\vec{n} \cdot \vec{X} = \vec{n} \cdot \vec{X_0} \longrightarrow \boxed{2x + y - z = 0}$$

8. *(12 pts)* Determine if the vectors $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ are linearly dependent or linearly independent.

Need to know if $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has any free variables.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} & \begin{array}{l} (1) \\ (2)-(1) \\ (3) \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} & \begin{array}{l} (1)+(2) \\ (2) \\ (3)+(2) \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} (1) \\ -(2) \\ (3)/2 \end{array} \end{array}$$

$$\longrightarrow \begin{array}{cl} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} (1)-(3) \\ (2)+(3) \\ (3) \end{array} \end{array}$$

There is a pivot in every column, so no free variables.

$\Rightarrow$ no non-trivial sol'ns to $A\vec{c} = \vec{0}$

$\Rightarrow$ no non-trivial relations

$\Rightarrow$ vectors are <u>linearly independent</u>

9. *(10 pts)* Bob says that the following system of equations can be used to view T and K as endogenous variables determined by the other variables. Is he correct? Explain why or why not.

$$\begin{array}{rcrcrcl} 4T & + & 4M & + & 2K & = & 2+U \\ 5T & + & z^2M & + & 5K & = & xy-z \end{array}$$

We can rewrite the system as

$$4T + 2K = -4M + 2 + U$$
$$5T + 5K = -z^2M + xy - z$$

The coefficient matrix $\begin{pmatrix} 4 & 2 \\ 5 & 5 \end{pmatrix}$ has determinant equal to $10 \neq 0$, so we do get uniqueness & existence.

Bob is right.

10. *(12 pts)* Prove that a linearly independent set of three vectors in $\mathbb{R}^3$ must span $\mathbb{R}^3$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ l. i. $\Rightarrow \left(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \Rightarrow (c_1, c_2, c_3) = \vec{0}\right)$

$\Rightarrow \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \vec{c} = \vec{0}$ has unique sol'ns

$\Rightarrow$ matrix has piv. in every col.

$\Rightarrow$ " " " " " " " " row,

$\Rightarrow \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \vec{c} = \vec{b}$ has existence

$\Rightarrow$ Every $\vec{b} \in \mathbb{R}^3$ is a l.c. of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^3$.