EXAM 2

Math 102, Spring 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________

1. __________
2. __________
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6. __________
7. __________
8. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

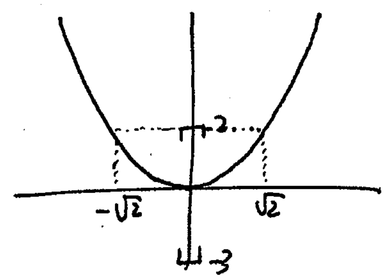
Total Score __________ (/100 points)

1. *(12 pts)* In this problem, we consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$.

(a) Determine $f(f^{-1}([-3, 2]))$.

$$f^{-1}([-3,2]) = [-\sqrt{2}, \sqrt{2}]$$

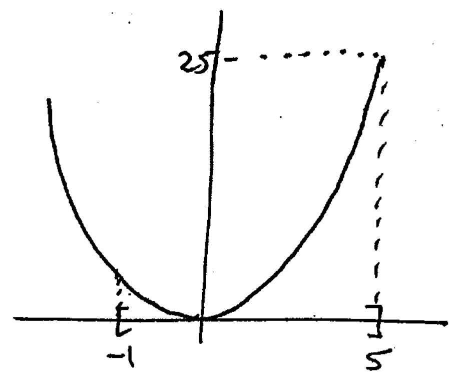
$$f([-\sqrt{2}, \sqrt{2}]) = \boxed{[0,2]}$$



(b) Determine $f^{-1}(f([-1, 5]))$.

$$f([-1,5]) = [0,25]$$

$$f^{-1}([0,25]) = \boxed{[-5,5]}$$



2. *(12 pts)* Which of the level sets of the function $f(x, y) = (x^2 - y)^2$ can be viewed as the graph of some other function?

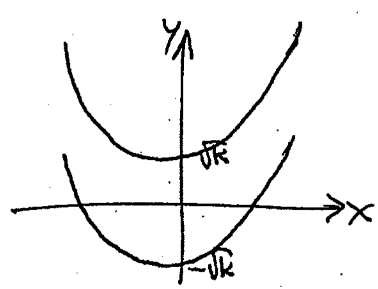
Level sets of f are $(x^2-y)^2 = k$

1. If $k<0$, these are empty.
2. If $k=0$, $(x^2-y)^2=0 \iff x^2-y=0 \iff y=x^2$ $\boxed{\underline{\text{graph}} \text{ of } g(x)=x^2}$
3. If $k>0$, $(x^2-y)^2=k$

$$\iff x^2-y = \pm\sqrt{k}$$

$$\iff y = x^2 \mp \sqrt{k}$$

<u>not</u> graphs, fail vertical line test.



3. *(12 pts)* Suppose that S is the graph of the function $f(x,y) = x^3 + y^2e^{(x^2-y^2-3)}$, P is the plane $x = 2$, C is the curve that is the intersection of S and P, and L is the line that is tangent to C at the point $(2,1,9)$. What is the slope of the line L?

We want the slope of the tangent line (L) to the cross section (c) of the graph (s) of f, in the y-direction.

So, $m = \frac{\partial f}{\partial y}(2,1)$.

$$\frac{\partial f}{\partial y} = (2y)e^{x^2-y^2-3} + (y^2)\left(e^{x^2-y^2-3}\right)(-2y)$$

$$\frac{\partial f}{\partial y}(2,1) = \boxed{0}$$

4. *(12 pts)* Suppose that the profit function for your company depends on costs of materials (C), capital (K), and labor (L), as indicated by the equation

$$P(C,K,L) = K^{1/2}L^{1/2}(10-C) \qquad \vec{x}^*$$

Currently we have $C = 4$, $K = 25$, and $L = 64$. If these inputs change to values $C = 4.1$, $K = 25.2$, and $L = 63.9$, what is the differential estimate of the change to the company's profits?

$$\frac{\partial P}{\partial C} = -K^{1/2}L^{1/2} \qquad \frac{\partial P}{\partial C}(\vec{x}^*) = -5 \cdot 8 = -40$$

$$\frac{\partial P}{\partial K} = \frac{1}{2}K^{-1/2}L^{1/2}(10-C) \qquad \frac{\partial P}{\partial K}(\vec{x}^*) = \frac{1}{2}\cdot\frac{1}{5}\,8\cdot 6 = \frac{48}{10} = \frac{24}{5}$$

$$\frac{\partial P}{\partial L} = \frac{1}{2}K^{1/2}L^{-1/2}(10-C) \qquad \frac{\partial P}{\partial L}(\vec{x}^*) = \frac{1}{2}\cdot 5\cdot\frac{1}{8}\cdot 6 = \frac{15}{8}$$

$$dP = \frac{\partial P}{\partial C}dC + \frac{\partial P}{\partial K}dK + \frac{\partial P}{\partial L}dL = (-40)(.1) + \left(\frac{24}{5}\right)(.2) + \left(\frac{15}{8}\right)(-.1)$$

$$= -4 + .96 - .1875 = -3.2275$$

5. *(14 pts)* Suppose that w is a function of x, y and z, and also that $x(s,t) = s - t$, $y(s,t) = st - t^2$, $z(s,t) = t^3 - s$. Suppose further that we are given that

$$\frac{\partial w}{\partial x}(2,4,4) = 3 \quad \text{and} \quad \frac{\partial w}{\partial z}(2,4,4) = -2$$

Compute $\frac{\partial w}{\partial t}(4,2)$.

$$\frac{\partial x}{\partial t} = -1 \qquad \frac{\partial x}{\partial t}(4,2) = -1$$

$$\frac{\partial y}{\partial t} = s - 2t \qquad \frac{\partial y}{\partial t}(4,2) = 0$$

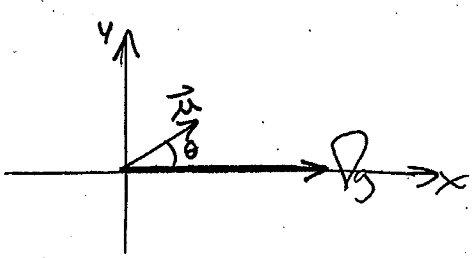
$$\frac{\partial z}{\partial t} = 3t^2 \qquad \frac{\partial z}{\partial t}(4,2) = 12$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (3)(-1) + \left(\frac{\partial w}{\partial y}\right)(0) + (-2)(12)$$

$$= \boxed{-27}$$

6. *(12 pts)* At the point $\vec{x}^*$, the function g is known to have gradient vector given by $\nabla g = (12, 0)$. At what angle from horizontal is it the case that the directional derivative in that direction is 6?



$$6 = D_{\vec{u}}g = \nabla g \cdot \vec{u}$$

$$= \|\nabla g\| \, \|\vec{u}\| \cos\theta$$

$$= 12 \cos\theta$$

$$\Rightarrow \cos\theta = 1/2$$

$$\Rightarrow \boxed{\theta = \pi/3}$$

7. *(14 pts)* We know that the variables v, w, x, y, and z are related by the equation $v^2z - xy + e^{w^2xz} = 4$. Compute the value of $\frac{\partial x}{\partial v}$ at the point $\vec{x}^* = (1, 2, 0, 5, 3)$, if it exists.

① We check if x is a fn of other variables by looking at $\frac{\partial F}{\partial x}$:

$$\frac{\partial F}{\partial x} = -y + \left(e^{w^2xz}\right)\left(w^2z\right) \qquad \frac{\partial F}{\partial x}(\vec{x}^*) = -5 + (1)(12) = 7$$

$\neq 0 \Rightarrow$ w is a fn of other vars

② Write partial of above wrt. v:

$$2vz - y\frac{\partial x}{\partial v} + \left(e^{w^2xz}\right)\left(w^2z\frac{\partial x}{\partial v}\right) = 0$$

$$\frac{\partial x}{\partial v} = \frac{2vz}{y - w^2ze^{w^2xz}}$$

$$\frac{\partial x}{\partial v}(\vec{x}^*) = \frac{6}{5 - 12\cdot 1} = \boxed{-\frac{6}{7}}$$

8. *(12 pts)* Find the unique vector $\vec{n}$ that is perpendicular to the surface with equation $x^3y^2 + 14 = yz^2$ at the point (1,2,3), and whose third coordinate is 24.

Surface is a l.s. of $g(x,y,z) = x^3y^2 - yz^2 + 14 = 0$

So ∇g is a normal vector:

$$\nabla g = \begin{pmatrix} 3x^2y^2 \\ 2x^3y - z^2 \\ -2yz \end{pmatrix} \qquad \nabla g(1,2,3) = \begin{pmatrix} 12 \\ -5 \\ -12 \end{pmatrix}$$

We know $\vec{n}$ must be parallel to this, so

$$\vec{n} = \begin{pmatrix} a \\ b \\ 24 \end{pmatrix} = k\begin{pmatrix} 12 \\ -5 \\ -12 \end{pmatrix} \qquad \Rightarrow k = -2$$

$$\Rightarrow \boxed{\vec{n} = \begin{pmatrix} -24 \\ 10 \\ 24 \end{pmatrix}}$$