EXAM 3

Math 102, Fall 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Use the Lagrange method to find the absolute maximum value of the function $f(x,y) = y - 3x^2$ on the domain defined by $y^2 - x^4 = 0$. *(You may assume for this problem the true fact that an absolute maximum value for this function on this domain is attained somewhere on this domain.)*

Degen.: Need $\nabla g = 0$.

$$\nabla g = \begin{pmatrix} -4x^3 \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x=0, y=0$$

crit. pt.

Lagr.: $\begin{pmatrix} -6x \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} -4x^3 \\ 2y \end{pmatrix}$

2nd eq $\Rightarrow \lambda = \frac{1}{2y}$ and $y \neq 0$

1st eq $\Rightarrow -6x = \left(\frac{1}{2y}\right)\left(-4x^3\right)$

$\Rightarrow y = \frac{1}{3}x^2$

$\Rightarrow \left(\frac{1}{3}x^2\right)^2 - x^4 = 0$

$\Rightarrow x = 0 \Rightarrow y = 0$

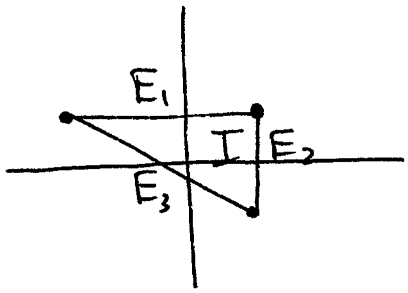
not a solution because

Only 1 critical point, at $(0,0)$.

So this is the only candidate for a maximum.

Given that the maximum exists, this must be it.

2. *(20 pts)* Find the absolute maximum value of the function $f(x, y) = x^2 + y^2$ on the solid triangle with vertices at the points $(-2, 1)$, $(1, -1)$ and $(1, 1)$.



On I: 0 local constraints,

$$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = 0$$

$\Rightarrow x=0, y=0 \leftarrow$ crit. pt.

On E_1: 1 local constraint, $y=1$ g_1

Deg: $\nabla g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Lagr: $\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow x=0$ $\quad (0,1)$ is crit. pt.

On E_2: 1 local constraint, $g_2 = x = 1$

Deg: $\nabla g_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Lagr: $\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow y=0$ $\quad (1,0)$ is crit. pt.

On E_3: 1 local constraint, $g_3 = 2x+3y = -1$

Deg: $\nabla g_3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Lagr: $\begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda_3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow 3x = 2y$

$6x = 4y$

Constr.: $6x + 9y = -3$

$4y + 9y = -3 \Rightarrow y = \frac{-3}{13}, \; x = -\frac{2}{13}$ $\Big\}$ crit. pt.

All 3 verts. are crit. pts.

Check values:

$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ $\qquad f\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 5 \leftarrow$ abs. max.

$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$ $\qquad f\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2$

$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ $\qquad f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$

$f\begin{pmatrix} -2/13 \\ -3/13 \end{pmatrix} = \frac{1}{13}$

3. *(20 pts)* Find the absolute maximum value of the function $f(x, y, z) = x + 2y + 3z$ with the constraints that $y^2 + z^2 = 1$ and $x + y + z = 0$.

$$\nabla f = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Dh_1 = \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix} \quad Dh_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Degen: Need $Dh = \begin{pmatrix} 0 & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$ to have rank < 2

$\Rightarrow y = 0,\ z = 0$; not on $y^2 + z^2 = 1$. $\times$

Lagr.: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

1st eq. $\Rightarrow \lambda_2 = 1$

2nd, 3rd $\Rightarrow$ $\begin{array}{l} 2 = \lambda_1 \cdot 2y + 1 \\ 3 = \lambda_1 \cdot 2z + 1 \end{array}$ $\Rightarrow$ $\begin{array}{l} 1 = \lambda_1 \cdot 2y \\ 2 = \lambda_1 \cdot 2z \end{array}$

$\Rightarrow z = 2y$

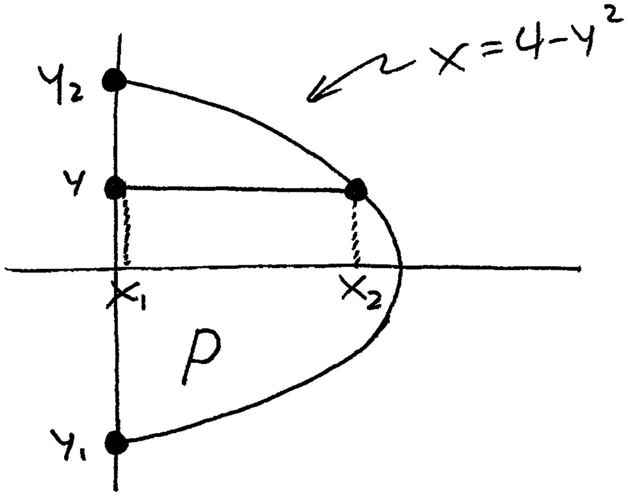
h_1 constr. $\Rightarrow y^2 + (2y)^2 = 1 \Rightarrow y = \pm\sqrt{1/5}$

$\Rightarrow$ 2 c.p.'s: $\begin{pmatrix} -3\sqrt{1/5} \\ \sqrt{1/5} \\ 2\sqrt{1/5} \end{pmatrix}, \begin{pmatrix} 3\sqrt{1/5} \\ -\sqrt{1/5} \\ -2\sqrt{1/5} \end{pmatrix}$

$f = 5\sqrt{1/5} = \sqrt{5}$ $\qquad f = -\sqrt{5}$

Max is $\boxed{f\begin{pmatrix} -3\sqrt{1/5} \\ \sqrt{1/5} \\ 2\sqrt{1/5} \end{pmatrix} = \sqrt{5}}$

4. *(20 pts)* The population density of Parabol county is given by $\delta(x, y) = 3 + x + y$. The county borders are the y-axis and the curve $x + y^2 = 4$. Write down a double integral representing the total population of Parabol county, rewrite it as a nested integral, and then evaluate.



Slice $\perp$ y-axis:

$y \in [y_1, y_2]$

$x=0$ and $x + y^2 = 4$ for each,

$\Rightarrow y_i = \pm 2$

For fixed $y \in [-2, 2]$, $x \in [x_1, x_2]$

$x_1 = 0$ from figure.

x_2 is on $x = 4 - y^2 \Rightarrow x_2 = 4 - y^2$

$$\text{Population} = \iint_P \delta(x,y)\, dA = \iint_P 3 + x + y \, dA$$

$$= \int_{-2}^{2} \int_0^{4-y^2} 3 + x + y \; dx \, dy$$

$$= \int_{-2}^{2} \left(3x + \tfrac{1}{2}x^2 + xy \right]_{x=0}^{x=4-y^2} dy$$

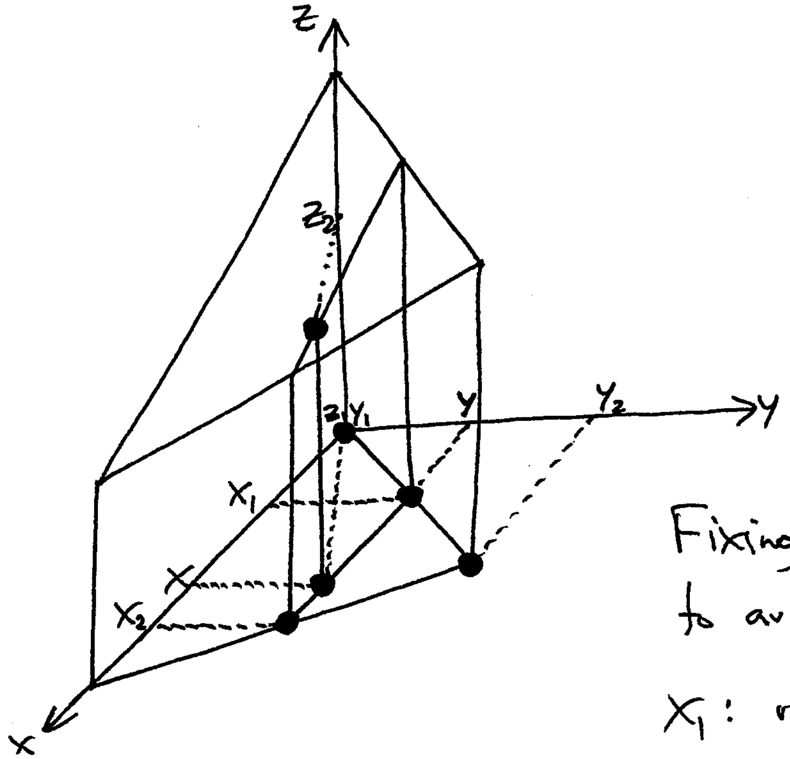
$$= \int_{-2}^{2} \left(3(4-y^2) + \tfrac{1}{2}(16 - 8y^2 + y^4) + (4-y^2)y \right) - 0 \; dy$$

$$= \int_{-2}^{2} 20 + 4y - 7y^2 - y^3 + \tfrac{1}{2}y^4 \, dy$$

$$= 20y + 2y^2 - \tfrac{7}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{10}y^5 \Big]_{-2}^{2} = 80 - \tfrac{112}{3} + \tfrac{64}{10}$$

49.0667

5. *(20 pts)* Write the triple integral $\iiint_D f\,dV$ as a single explicit nested integral, but do not evaluate. The domain D is bounded by the surfaces $y = 0$, $x = y$, $x + y = 2$, $z = 0$, $z = 4 - x$.



Must slice $\perp$ y-axis first to avoid corners.

$y \in [y_1, y_2]$

y_1: on $y=0$, so $y_1 = 0$

y_2: on $x=y$, $x+y=2$, so $y_2 = 1$

Fixing y, must slice $\perp$ x-axis first to avoid corners. $x \in [x_1, x_2]$

x_1: on $x=y$, $y=y$, so $x_1 = y$

x_2: on $x+y=2$, $y=y$, so $x_2 = 2-y$

Fixing x, we have $z \in [z_1, z_2]$

z_1: on $z=0$, so $z_1 = 0$

z_2: on $z=4-x$, so $z_2 = 4-x$

So $$\iiint_D f\,dV = \int_0^1 \int_y^{2-y} \int_0^{4-x} f \, dz\,dx\,dy$$