EXAM 1

Math 102, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
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6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(15 pts)*

Find the complete set of solutions to the system of equations below.

$$\begin{array}{rcrcrcr} 1x & + & 2y & + & 6z & = & -8 \\ -1x & - & 2y & - & 4z & = & 4 \\ 2x & + & 4y & + & 10z & = & -12 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 6 & -8 \\ -1 & -2 & -4 & 4 \\ 2 & 4 & 10 & -12 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 6 & -8 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -2 & 4 \end{array}\right) \begin{array}{l} (1) \\ (2)+(1) \\ (3)-2(1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} (1)-3(2) \\ (2) \\ (3)+(2) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} (1) \\ (2)/2 \\ (3) \end{array}$$

$\rightarrow$ $z=-2$, $x+2y=4$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-2y \\ y \\ -2 \end{pmatrix}$

2. *(10 pts)* Let A denote the (non-augmented) matrix of coefficients for the above system. Use the row reduction process from the above problem to compute the determinant of A.

The above process is summarized by

$$E_3 E_2 E_1 A = R$$

$$\Rightarrow (\det E_3)(\det E_2)(\det E_1)(\det A) = \det R$$

$$\Rightarrow \tfrac{1}{2} \cdot 1 \cdot 1 \cdot \det A = 0$$

$$\Rightarrow \boxed{\det A = 0}$$

3. *(10 pts)* Suppose we wish to find a vector $\vec{b}$ for which the system $A\vec{x} = \vec{b}$ has no solutions. Write such a vector as a product of elementary matrices and another vector $\vec{c}$, but do not multiply out the product. Make sure to explain exactly where you got those elementary matrices and how you chose the vector $\vec{c}$.

The elementary matrices denoted in (2) are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reducing the system $A\vec{x} = \vec{b}$ we get

$$E_3 E_2 E_1 A \vec{x} = E_3 E_2 E_1 \vec{b}$$

$$\Rightarrow R\vec{x} = E_3 E_2 E_1 \vec{b} = \vec{c}$$

To ensure there are no solutions, we put a nonzero third coordinate for $\vec{c}$; choose $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $E_3 E_2 E_1 \vec{b} = \vec{c}$

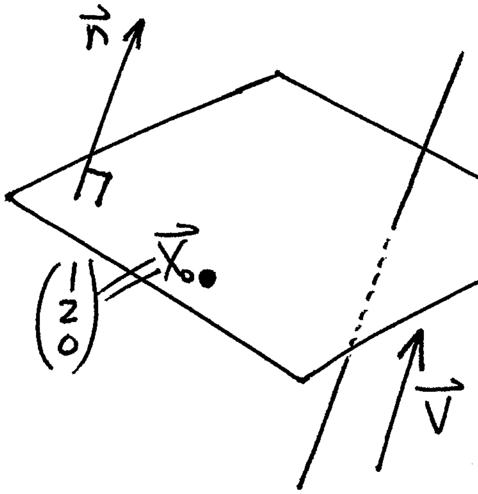
$$\vec{b} = E_1^{-1} E_2^{-1} E_3^{-1} \vec{c}$$

So

$$\vec{b} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4. *(15 pts)* Find the equation of the unique plane that contains the point $(1, 2, 0)$ and is perpendicular to the line described parametrically by

$$\vec{x}(t) = (3 - 4t, 2 + 3t, t - 2) = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} + t \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$$



$$\vec{V} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$$

$\vec{V} \perp$ plane, $\vec{n} \perp$ plane, so we can choose $\vec{n} = \vec{V}$.

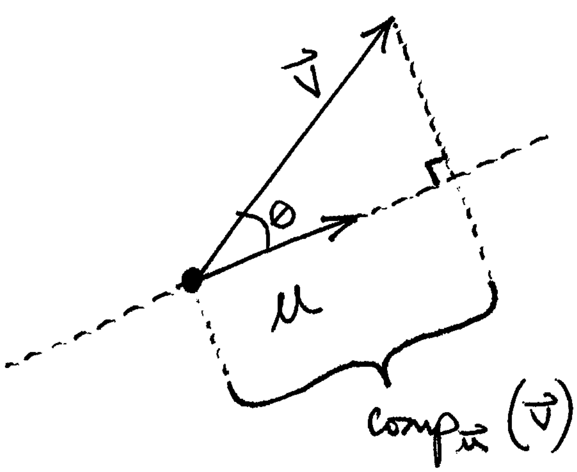
Eq. is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

$$\boxed{-4x + 3y + z = 2}$$

5. *(20 pts)* We define the "component of $\vec{v}$ along $\vec{w}$" (written "$\text{comp}_{\vec{w}}(\vec{v})$") to be the length of the projection of the vector $\vec{v}$ onto the line pointing in the direction of the vector $\vec{w}$. Show that if $\vec{u}$ is a unit vector, then

$$\text{comp}_{\vec{u}}(\vec{v}) = \vec{u} \cdot \vec{v}$$

(Hint: First write this component with trig in terms of the angle between the two vectors.)



Trig gives us

$$\text{comp}_{\vec{u}}(\vec{v}) = \|\vec{v}\| \cos\theta$$

Since $\|\vec{u}\| = 1$, we can rewrite this as

$$\text{comp}_{\vec{u}}(\vec{v}) = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$
$$= \vec{u} \cdot \vec{v}$$

6. *(15 pts)* Suppose we have

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 6 \\ 3 & 5 \end{pmatrix}$$

Compute the following.

(a) AB

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 3 & 5 \end{pmatrix} = \boxed{\begin{pmatrix} 24 & 42 \\ 12 & 21 \end{pmatrix}}$$

(b) rank(A)

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{matrix} (1)/2 \\ (2) \end{matrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{matrix} (1) \\ (2)-(1) \end{matrix}$$

There is one pivot in rref(A), so

$$\boxed{\text{rank}(A) = 1}$$

(c) B^{-1}

$$\left(\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \begin{matrix} (1)/3 \\ (2) \end{matrix}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1/3 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \begin{matrix} (1) \\ (2)-3(1) \end{matrix}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -5/3 & 2 \\ 0 & -1 & -1 & 1 \end{array}\right) \begin{matrix} (1)+2(2) \\ (2) \end{matrix}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -5/3 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right) \begin{matrix} (1) \\ -(2) \end{matrix}$$

So

$$\boxed{B^{-1} = \begin{pmatrix} -5/3 & 2 \\ 1 & -1 \end{pmatrix}}$$

7. *(15 pts)* Suppose that $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are vectors in $\mathbb{R}^5$, that A is the matrix whose rows are these three vectors, and that

$$\begin{pmatrix} 6 & 9 & 0 \\ 23 & 0 & -21 \\ 0 & -12 & 14 \end{pmatrix} \begin{pmatrix} & & A & & \end{pmatrix} = \begin{pmatrix} 7 & 8 & 0 & -31 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -42 & 0 & -82 & 0 & 15 \end{pmatrix}$$

Can you conclude from this information either that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent or linearly dependent? Make sure to justify your answer explicitly.

We use the fact that the rows of MA are linear combinations of the rows of A, where corresponding rows of M are the coefficients.

Applying this to the second row of the above product, we have:

$$23\vec{v}_1 + 0\vec{v}_2 - 21\vec{v}_3 = \vec{0}$$

This is a nontrivial relation, so

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a <u>linearly dependent</u> set.