EXAM 2

Math 102, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number______________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

Total Score __________ (/100 points)

1. *(14 pts)* Determine if the set of vectors below forms a basis for $\mathbb{R}^3$:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0$$

So $\{\vec{u}, \vec{v}, \vec{w}\}$ is an independent set of three vectors in $\mathbb{R}^3$, so they form a basis for $\mathbb{R}^3$.

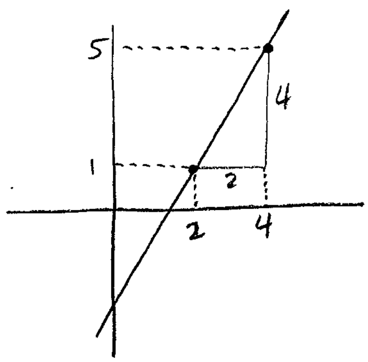
2. *(14 pts)* In what unit vector direction is the function $f(x, y, z) = x^2y - z^3 + 4xz$ increasing most quickly from the point $(1, -1, 2)$?

$$\nabla f = \begin{pmatrix} 2xy + 4z \\ x^2 \\ -3z^2 + 4x \end{pmatrix} \qquad \nabla f\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$$

$$\vec{u} = \frac{\nabla f}{\|\nabla f\|} = \frac{\nabla f}{\sqrt{101}} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} \Big/ \sqrt{101}$$

3. *(24 pts)* In this problem we consider the curve C that is the unique line passing through the points $(2,1)$ and $(4,5)$.

(a) Find a function $f : \mathbb{R}^a \to \mathbb{R}^b$ such that C is the graph of f, and make sure to identify explicitly the values of a and b.



$$y-1 = (2)(x-2) \Rightarrow y = 2x-3$$

This is the graph of $f : \mathbb{R}^1 \to \mathbb{R}^1$ given by $f(x) = 2x-3$

(b) Find a function $g : \mathbb{R}^c \to \mathbb{R}^d$ such that C is a level set of g, and make sure to identify explicitly the values of c and d.

$$y = 2x-3 \Leftrightarrow y-2x = 3$$

This is the equation for the $g=3$ level set of $g : \mathbb{R}^2 \to \mathbb{R}^1$ given by $g(x,y) = y-2x$

(c) Find a function $h : \mathbb{R}^i \to \mathbb{R}^j$ that represents C parametrically, and make sure to identify explicitly the values of i and j.

$\vec{x}_0 = (2,1)$

$\vec{v} = (4,5) - (2,1)$

$= (2,4)$

$\vec{x} = (2,1) + t(2,4)$

C is parametrized by $h : \mathbb{R}^1 \to \mathbb{R}^2$ given by $h(t) = (2,1) + t(2,4)$

4. *(14 pts)* At the company where you work, the expected profit (P) from production with input parameters r, s, and t is given by

$$P = \left(e^{6-r^2-s^2-t^2}\right)(r^2 s)$$

(2,1,1)

At the moment the company has the parameters set at $(r^*, s^*, t^*) =$ [redacted]. You have heard through the grapevine that the company will be changing s and t by amounts $ds = .01$ and $dt = .02$, and also that the company expects to increase its profits by $dP = .24$. What can you infer about planned changes to r?

$$\frac{\partial P}{\partial r} = \left(-2r\, e^{6-r^2-s^2-t^2}\right)\left(r^2 s\right) + \left(e^{6-r^2-s^2-t^2}\right)(2rs)$$

$$= \left(2rs - 2r^3 s\right) e^{6-r^2-s^2-t^2}$$

$$\frac{\partial P}{\partial s} = \left(-2s\, e^{6-r^2-s^2-t^2}\right)\left(r^2 s\right) + \left(e^{6-r^2-s^2-t^2}\right)\left(r^2\right)$$

$$= \left(r^2 - 2r^2 s^2\right) e^{6-r^2-s^2-t^2}$$

$$\frac{\partial P}{\partial t} = \left(r^2 s\right)(-2t)\, e^{6-r^2-s^2-t^2}$$

At the given point, we have

$$\frac{\partial P}{\partial r} = -12 \qquad \frac{\partial P}{\partial s} = -4 \qquad \frac{\partial P}{\partial t} = -8$$

The total derivative of P then is

$$dP = -12\,dr - 4\,ds - 8\,dt$$

$$.24 = (-12)(dr) - (4)(.01) - (8)(.02)$$

$$.44 = -12\,dr \implies dr = -\frac{.44}{12}$$

5. *(14 pts)* Consider the functions below:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 - x_2 \\ x_2^2 - x_1 \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 x_2 \end{bmatrix}$$

and the composition defined by

$$h(\vec{x}) = f\left(g(\vec{x})\right)$$

Use the chain rule to...
Find the derivative matrix Dh at the point $(2, 1)$.

To avoid confusion of variables we write

$$f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 - y_2 \\ y_2^2 - y_1 \end{pmatrix}$$

Then

$$\begin{tikzcd} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \arrow[r, "g"] \arrow[rr, bend right, "h"'] & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \arrow[r, "f"] & \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{tikzcd}$$

And $Dh = (Df)(Dg)$

$$= \begin{pmatrix} 2y_1 & -1 \\ -1 & 2y_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ x_2 & x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 2y_1 - x_2 & -2y_1 - x_1 \\ -1 + 2x_2 y_2 & 1 + 2x_1 y_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(x_1 - x_2) - x_2 & -2(x_1 - x_2) - x_1 \\ -1 + 2x_2(x_1 x_2) & 1 + 2x_1(x_1 x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 - 3x_2 & -3x_1 + 2x_2 \\ -1 + 2x_1 x_2^2 & 1 + 2x_1^2 x_2 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 3 & 9 \end{pmatrix}$$

6. *(20 pts)* Consider the system

$$\begin{aligned} u^2 - uv^2 + vx - uy^2 + yz &= 1 \\ u + v - xy + 2yz &= 8 \end{aligned}$$

near the point $(u^*, v^*, x^*, y^*, z^*) = (1, 2, 1, 1, 3)$. Determine if the variables x and y can be viewed locally as implicit functions of the other variables. If they can, compute their partial derivatives with respect to u at the given point.

$$D_{xy}F = \begin{pmatrix} v & -2uy + z \\ -y & -x + 2z \end{pmatrix}$$

$$D_{xy}F(1,2,1,1,3) = \begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}$$

$\det D_{xy}\vec{F} = 11 \neq 0$

So x, y can be viewed locally as implicit fns.

Differentiating the above system w.r.t. u then, we get

$$(D_{xy}F)\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} + \begin{pmatrix} \partial F_1/\partial u \\ \partial F_2/\partial u \end{pmatrix} = \vec{0}$$

$$(D_{xy}F)\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} + \begin{pmatrix} 2u - v^2 - y^2 \\ 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \left(\frac{1}{11}\right)\begin{pmatrix} 5 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 16/11 \\ 1/11 \end{pmatrix}$$