EXAM 3

Math 102, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
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5. __________
6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(10 pts)* Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Is f locally invertible at every point in its domain? If not, find all of the counterexamples.

$$Df = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$$\det(Df) = e^{2x}\cos^2 y - (-e^{2x}\sin^2 y) = e^{2x}$$

This is never zero, so f is locally invertible everywhere in the domain.

2. *(10 pts)* Determine if the following matrix is positive (or negative) definite (or semidefinite), or indefinite:

$$A = \begin{pmatrix} 5 & 0 & 7 & 0 \\ 0 & 0 & 3 & 2 \\ 7 & 3 & 0 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

For $Q(\vec{x}) = \vec{x} \cdot A\vec{x}$, we have

$$Q(\vec{e}_1) = 5$$
$$Q(\vec{e}_4) = -1$$

So Q is $\boxed{\text{indefinite.}}$

3. *(16 pts)* Find the absolute maximum value of the function $f(x,y) = 32x - x^4 + 108y - y^4$ on $\mathbb{R}^2$.

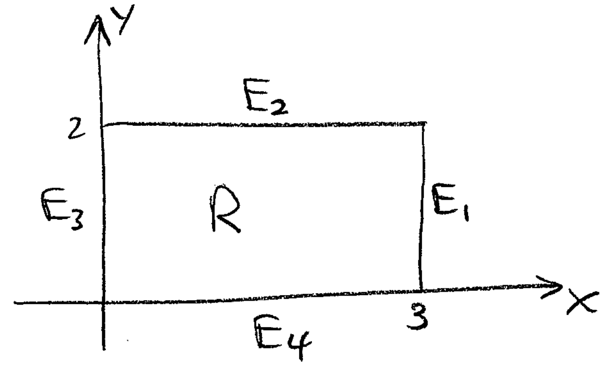
$$\nabla f = \begin{pmatrix} 32 - 4x^3 \\ 108 - 4y^3 \end{pmatrix} = \vec{0} \quad \Rightarrow \quad \begin{array}{l} x = 2 \\ y = 3 \end{array}$$

$$H = \begin{pmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix}$$

This matrix is negative semidefinite for all x, y ; $\mathbb{R}^2$ is open and convex.

So, the above critical point is the absolute maximum.

4. *(16 pts)* Find the absolute maximum value of the function $f(x,y) = x^3 + 3y^3 - 12x + 7y$ on the solid rectangle R in the xy-plane with corners at $(0,0)$, $(3,0)$, $(0,2)$, $(3,2)$.



Interior: $\nabla f = \begin{pmatrix} 3x^2 - 12 \\ 9y^2 + 7 \end{pmatrix} = \vec{0}$

$9y^2 + 7 \neq 0$, so no int. c.p.'s.

Edge 1: $h_1 = x = 3$, $\nabla h_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$ non-degen. $\times$

$\nabla f = \mu_1 \nabla h_1 \Rightarrow \begin{pmatrix} 3x^2-12 \\ 9y^2+7 \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow 9y^2 + 7 = 0 \times$

Edge 2: $h_2 = y = 2$, $\nabla h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow$ non-deg. $\times$

$\nabla f = \mu_2 \nabla h_2 \Rightarrow \begin{pmatrix} 3x^2-12 \\ 9y^2+7 \end{pmatrix} = \mu_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow x = 2, y = 2$

Edge 3: $h_1 = x = 0$, $\nabla h_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$ non-deg. $\times$

$\nabla f = \mu_1 \nabla h_1 \Rightarrow 9y^2 + 7 = 0 \times$

Edge 4: $h_2 = y = 0$, $\nabla h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftarrow$ non-deg. $\times$

$\nabla f = \mu_2 \nabla h_2 \Rightarrow x = 2, y = 0$

Six total c.p.'s: 1 on E_2, 1 on E_4, 4 at vertices

$f(0,0) = 0$ $\quad$ $f(2,2) = 22$

$f(3,0) = -9$ $\quad$ $f(2,0) = -16$

$f(0,2) = 38 \leftarrow$ abs. max

$f(3,2) = 27$

5. *(16 pts)* Find the absolute maximum value of the function $f(x, y, z) = z$ with constraints $h_1 = z - x^3 + 6y^2 = 0$ and $h_2 = x^2 + y^2 = 25$.

$$\nabla f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \nabla h_1 = \begin{pmatrix} -3x^2 \\ 12y \\ 1 \end{pmatrix} \quad \nabla h_2 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$$

<u>Degen</u>: $Dh = \begin{pmatrix} -3x^2 & 12y & 1 \\ 2x & 2y & 0 \end{pmatrix}$ to be degen, need (bottom) = 0 · (top)

$\Rightarrow x = y = 0$, not possible with $x^2+y^2=25$ ✗

<u>Lagr</u>: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mu_1 \begin{pmatrix} -3x^2 \\ 12y \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$ $\Rightarrow \mu_1 = 1$

$$\Rightarrow \quad 0 = -3x^2 + (\mu_2)(2x)$$
$$0 = 12y + (\mu_2)(2y) \Rightarrow \mu_2 = -6 \text{ or } y = 0$$

If $y=0$, then $x = \pm 5$

If $\mu_2 = -6$, then $-3x^2 - 12x = 0$

$x = 0$, $y = \pm 5$ or $x = 4$, $y = \pm 3$

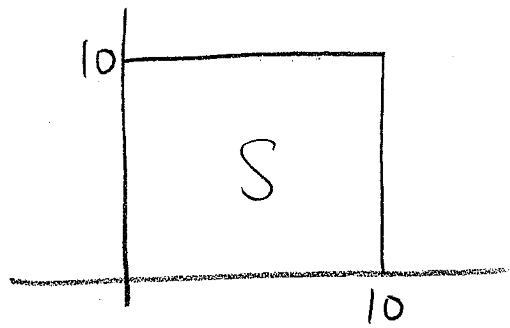
<u>6 c.p.'s:</u>

$$\begin{pmatrix} 5 \\ 0 \\ 125 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -125 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ -130 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ -130 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 10 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$$

abs. max.

6. *(16 pts)* Square County is in the shape of a square, with side length of 10 miles and the origin at the southwest corner of the county. The Sheriff of Square County has data that suggest that the density (in thousands of people per square mile) who support his candidacy for County Commissioner is given by $f(x,y) = xye^{-x^2}$. How many supporters can the Sheriff reasonably claim to have? (Feel free to use the approximation $e^{-100} \approx 0$.)

$$P = \iint_S \delta \, dA = \iint_S f(x,y)\, dx\, dy$$



$$= \int_0^{10}\int_0^{10} xye^{-x^2}\, dx\, dy$$

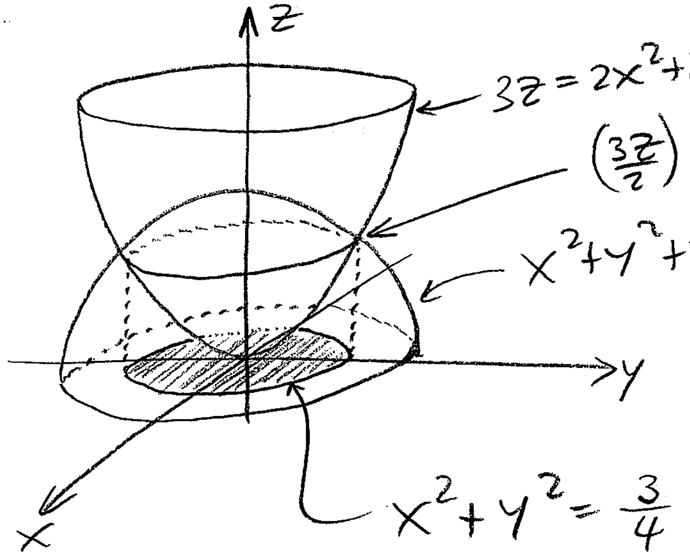
$$= \int_0^{10} \left(-\frac{1}{2} ye^{-x^2} \Big]_{x=0}^{x=10} \right) dy = \int_0^{10} \left(\left(0 \right) - \left(-\frac{1}{2} ye^{0} \right) \right) dy$$

$$= \int_0^{10} \frac{1}{2} y \, dy$$

$$= \frac{1}{4} y^2 \Big]_0^{10} = 25$$

The sheriff has about 25,000 supporters.

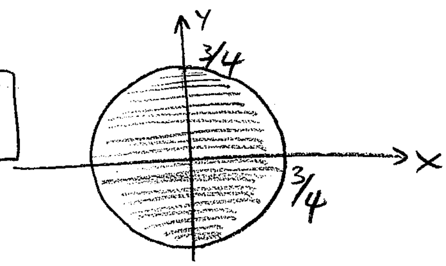
7. *(16 pts)* The region R is the region inside the unit sphere and also inside the paraboloid with equation $3z = 2x^2 + 2y^2$. Set up, but do not evaluate, a triple integral representing the mass of R where the density is given by $\delta(x, y, z) = e^{xyz}$.



$$\left(\frac{3z}{2}\right) + z^2 = 1 \Rightarrow 2z^2 + 3z - 2 = 0$$
$$(2z-1)(z+2) = 0$$
$$\Rightarrow z = \tfrac{1}{2}$$

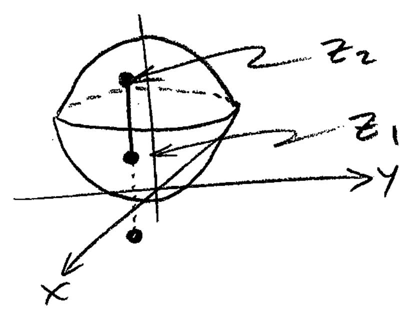
Looking at the projection to the xy-plane, we have

$x \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$, $y \in \left[-\sqrt{\frac{3}{4}-x^2}, \sqrt{\frac{3}{4}-x^2}\right]$,



For a given (x,y) then, we have

$z \in \left[\frac{2x^2+2y^2}{3}, \sqrt{1-x^2-y^2}\right]$



So

$$m = \iiint dm = \iiint \delta \, dV$$

$$\boxed{= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}} \int_{\frac{2x^2+2y^2}{3}}^{\sqrt{1-x^2-y^2}} e^{xyz} \, dz \, dy \, dx}$$