EXAM 1

Math 102, Fall 2009-2010, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(15 pts)* Determine whether the variables x, y, and z can be viewed as endogenous variables in the system below.

$$\begin{aligned} x - 4y + 2z &= M - E \\ 2x + y + 5z &= M^2 - EK \\ 5x - 2y + 12z &= EM \end{aligned}$$

$$\begin{pmatrix} 1 & -4 & 2 \\ 2 & 1 & 5 \\ 5 & -2 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -4 & 2 \\ 0 & 9 & 1 \\ 0 & 18 & 2 \end{pmatrix} \begin{array}{l} 1 \\ 2 - 2(1) \\ 3 - 5(1) \end{array}$$

$$\begin{pmatrix} 1 & -4 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} 1 \\ 2 \\ 3 - 2(2) \end{array}$$

It is clear at this point in the reduction that the original matrix is <u>not</u> non-singular. So the vars x, y, z cannot be viewed as endogenous.

2. *(10 pts)* For a given matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$, it is known that the system $A\vec{x} = \vec{b}_1$ has no solutions and that the system $A\vec{x} = \vec{b}_2$ has exactly one solution.

Your friends Carol and Dave are arguing about the determinant of A – Carol believes that because of the lack of existence in the first system, the determinant must be zero; Dave believes that because of the uniqueness in the second system, the determinant must be nonzero.

Settle this dispute and explain the problem with the invalid reasoning.

$A\vec{x} = \vec{b}_1$ has no solutions $\Rightarrow$ rank < # rows

$A\vec{x} = \vec{b}_2$ has uniqueness $\Rightarrow$ rank = # cols.

So # rows $\neq$ # cols, and matrix is <u>not square</u>.

So there is no determinant in the first place!

Both friends are wrong!

3. *(15 pts)* Use elementary matrices to show that if a matrix A is non-singular, then it must be invertible. *(Hint: Represent the row reduction of A with elementary matrices, and then think about what the reduced form must be.)*

Row reduction of A is: $E_k \cdots E_1 A = R$

Since A is non-singular, we have $R = I$.

Letting $E = E_k \cdots E_1$, we then have

$$EA = I$$

So E is the inverse of A, and so A is invertible.

4. *(15 pts)* The row reduction of the 4x4 matrix A proceeds by the following row operations (in each operation, the numbered row references are to the preceding matrix, not the original matrix).

   (a) The first and fourth rows are switched.

   (b) The first row is divided by 7.

   (c) 6 times the first row is added to the second row.

   (d) 4 times the first row is subtracted from the third row.

   (e) The second row is multiplied by 2.

   (f) 5 times the second row is added to the fourth row.

   (g) The third row is divided by 3.

   (h) 8 times the third row is subtracted from the fourth row.

   (i) The fourth row is divided by 5.

   The resulting reduced row echelon form matrix has a pivot in every column. What is the determinant of the original matrix A?

The original determinant is affected by the row ops by:

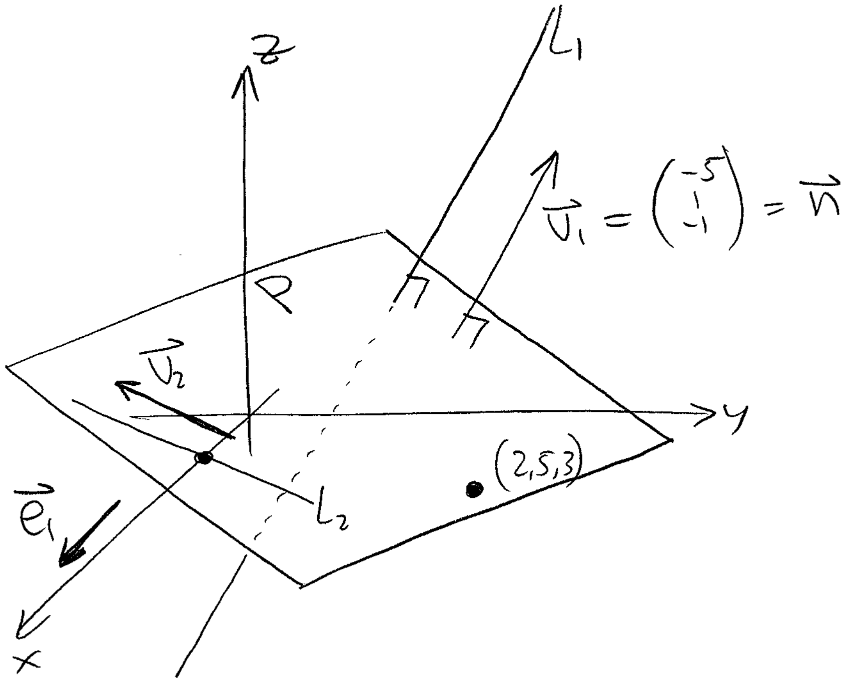
$$(\det A)(-1)\left(\frac{1}{7}\right)(2)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = 1$$

and equals 1 because with a pivot in every column the r.r.e.f. is the identity.

So $\det A = (-1)(7)\left(\frac{1}{2}\right)(3)(5) = \boxed{-\frac{105}{2}}$

5. *(15 pts)* The line L_1 is parametrized by $(3-5t, 2+t, 1-t)$. The plane P is perpendicular to L_1 and passes through the point $(2,5,3)$. The line L_2 is a subset of P, intersects the x-axis, and is perpendicular to the x-axis.

Find a parametric representation of the line L_2.



eqn of P: $\vec{n} \cdot \vec{x} = \vec{n} \cdot x_0$

$$-5x + y - z = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} = -8$$

x-int is on plane where $y = z = 0$:

$$-5x + 0 - 0 = -8 \quad \Rightarrow x = \frac{8}{5} \quad \Rightarrow \begin{pmatrix} 8/5 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}_2$ is $\perp \vec{n}$, and $\perp \vec{e}_1$, so we can choose

$$\vec{v}_2 = \vec{e}_1 \times \vec{n} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 0 & 0 \\ -5 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Line L_2 then is $\left\{ \begin{pmatrix} 8/5 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

6. *(15 pts)* Find a basis for the span of the set of vectors $\{(3,2,5),(1,1,5),(3,1,-5)\}$.

$$\det\begin{pmatrix}3 & 1 & 3\\ 2 & 1 & 1\\ 5 & 5 & -5\end{pmatrix} = (3)(-10)-(1)(-15)+(3)(5) = 0$$

So these three vectors are l.d., so the span has dimension of at most 2.

The vectors $\left\{\begin{pmatrix}3\\2\\5\end{pmatrix}, \begin{pmatrix}1\\1\\5\end{pmatrix}\right\}$ are not parallel and thus independent, so they are a basis for their span — which is thus of dimension 2, meaning also the original span must have dimension 2.

So $\boxed{\left\{\begin{pmatrix}3\\2\\5\end{pmatrix}, \begin{pmatrix}1\\1\\5\end{pmatrix}\right\}}$ are 2 l.i. vectors in the original 2-dim span, so they are a basis

7. *(15 pts)* The function $g:\mathbb{R}^3\to\mathbb{R}^1$ is given by $g(x,y,z)=(x^4+1)z-(y^2+4)e^x$. Find the function f (identify the domain, the target, and a formula for computing its values) whose graph is the level set of g that passes through the origin.

$g(0,0,0) = (0+1)(0) - (0+4)e^0 = -4$. So the level set of g passing through the origin has equation

$g(x,y,z) = -4$, or $(x^4+1)z-(y^2+4)e^x = -4$.

This is equivalent to $z = \dfrac{(y^2+4)e^x-4}{x^4+1}$

which is the graph of the function

$$\boxed{f:\mathbb{R}^2\to\mathbb{R}^1,\quad f(x,y)=\frac{(y^2+4)e^x-4}{x^4+1}}$$