EXAM 2

Math 102, Fall 2009-2010, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________

1. __________
2. __________
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6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________

Total Score __________ (/100 points)

1. *(15 pts)* Use the total derivative to estimate the value of $f(1.01, 0.02, 2.98)$, where the function f is defined by $f(x, y, z) = x^2 z e^{xyz} - x^2 yz$.

$\frac{\partial f}{\partial x} = 2xze^{xyz} + x^2 z(yz)e^{xyz} - 2xyz$ $\quad \frac{\partial f}{\partial x}(1,0,3) = 6$

$\frac{\partial f}{\partial y} = x^2 z(xz)e^{xyz} - x^2 z$ $\quad \frac{\partial f}{\partial y}(1,0,3) = 6$

$\frac{\partial f}{\partial z} = x^2 e^{xyz} + x^2 z(xy)e^{xyz} - x^2 y$ $\quad \frac{\partial f}{\partial z}(1,0,3) = 1$

$\vec{a} = (1,0,3)$, $\vec{x} = (1.01, 0.02, 2.98)$, $d\vec{x} = (.01, .02, -.02)$

$$f(\vec{x}) \approx f(\vec{a}) + df = 3 + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= 3 + (6)(.01) + (6)(.02) + (1)(-.02)$$

$$= \boxed{3.16}$$

2. *(10 pts)* Suppose we have $x = uv^2$, $y = u^2 - v^3$, $z = 3uv$, $r = xyz^2 - z^2$, $s = x^3 - y^3 - z^3$. Use the chain rule to compute the value of $\frac{\partial r}{\partial v}$ when $u = 1$ and $v = 2$.

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} r \\ s \end{pmatrix}$$

$$\frac{\partial r}{\partial v} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial v}$$

$$= (yz^2)(2uv) + (xz^2)(-3v^2) + (2xyz - 2z)(3u)$$

When $u=1, v=2$, we have $x = 4, y = -7, z = 6$

$$\frac{\partial r}{\partial v}(1,2) = (-7)(6)^2(2\cdot1\cdot2) + (4)(6)^2(-3\cdot 2^2) + (2\cdot4\cdot(-7)\cdot6 - 2\cdot6)(3\cdot1)$$

$$= -1008 - 1728 - 1044 = \boxed{-3780}$$

3. *(15 pts)* At the point $\vec{x}^* = (x^*, y^*, z^*)$, the differentiable function $g : \mathbb{R}^3 \to \mathbb{R}^1$ is increasing most quickly in the direction parallel to the vector $(-2, 6, -3)$, and the directional derivative in that direction is 35. What is the directional derivative of g at this point in the direction that is parallel to the vector $(12, 3, -4)$?

$\nabla g(\vec{x}^*) \parallel (-2,6,-3)$

Let $\vec{u} = \frac{(-2,6,-3)}{7}$ (unit vector)

and let $\nabla g = k\vec{u}$

$35 = D_{\vec{u}} g(\vec{x}^*) = \nabla g \cdot \vec{u}$

$= k\vec{u}\cdot\vec{u}$

$= k$

So $\nabla g = 35\vec{u}$

$= (-10, 30, -15)$

Let $\vec{w} = \frac{(12,3,-4)}{13}$ (unit vector)

$D_{\vec{w}} f(\vec{x}^*) = \nabla g \cdot \vec{w}$

$= \begin{pmatrix}-10\\30\\-15\end{pmatrix} \cdot \begin{pmatrix}12\\3\\-4\end{pmatrix} / 13 = \boxed{\frac{30}{13}}$

4. *(15 pts)* The variables w, x, y, and z are related by the equation $x^4y^2 - 2wxz + w^3yz = -2$. Compute $\frac{\partial w}{\partial x}$ at the point $(w^*, x^*, y^*, z^*) = (1, 2, 2, 3)$.

$F(w,x,y,z) = x^4y^2 - 2wxz + w^3yz = -2$

$\frac{\partial F}{\partial w} = -2xz + 3w^2yz$

$\frac{\partial F}{\partial w}(1,2,2,3) = -12 + 18 = 6 \neq 0$

So w can be viewed as a fn of x,y,z at this pt.

$\frac{\partial w}{\partial x}(1,2,2,3) = -\frac{\partial F/\partial x}{\partial F/\partial w}$

$= -\frac{4x^3y^2 - 2wz}{-2xz + 3w^2yz}$

$= -\frac{122}{6} = \boxed{-\frac{61}{3}}$

5. *(15 pts)* The variables p, q, r, s, and t are related by the system

$$\begin{aligned} F_1 = pq^2 - qr^2 - rs^2 + st^2 &= 4 \\ F_2 = p^2q + q^2r + r^2s - s^2t &= 2 \end{aligned}$$

Show that near the point $(p^*, q^*, r^*, s^*, t^*) = (1, 2, 0, 2, 0)$ we can view q and r as implicitly defined functions of the other variables, and compute $\frac{\partial r}{\partial p}$ at this point.

$$D_{qr}F = \begin{pmatrix} 2pq - r^2 & -2qr - s^2 \\ p^2 + 2qr & q^2 + 2rs \end{pmatrix}$$

$$D_{qr}F(1,2,0,2,0) = \begin{pmatrix} 4 & -4 \\ 1 & 4 \end{pmatrix} \leftarrow \det = 20 \neq 0$$

This det $\neq 0$, so q, r are locally functions of p, s, t.

Taking $\frac{\partial}{\partial p}$ of the system we have

$$\left(D_{qr}F\right)\begin{pmatrix} \partial q/\partial p \\ \partial r/\partial p \end{pmatrix} + \begin{pmatrix} \partial F_1/\partial p \\ \partial F_2/\partial p \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 4 & -4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \partial q/\partial p \\ \partial r/\partial p \end{pmatrix} + \begin{pmatrix} q^2 \\ 2pq \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 4 & -4 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} \partial q/\partial p \\ \partial r/\partial p \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

By Cramers rule $$\frac{\partial r}{\partial p} = \frac{\det\begin{pmatrix} 4 & -4 \\ 1 & -4 \end{pmatrix}}{\det\begin{pmatrix} 4 & -4 \\ 1 & 4 \end{pmatrix}} = \frac{-12}{20} = \boxed{\frac{-3}{5}}$$

6. *(15 pts)* Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (y, x^3 + xy)$. A what points in the domain is this function locally invertible?

$$Df = \begin{pmatrix} 0 & 1 \\ 3x^2+y & x \end{pmatrix} \qquad \det(Df) = -(3x^2+y)$$

f is locally invertible when Df is invertible, namely when $\det(Df) \neq 0$. This is when

$\boxed{3x^2+y \neq 0}$.

7. *(15 pts)* Let Q be the quadratic form defined by the matrix

$$A = \begin{pmatrix} -3 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Determine if this quadratic form is positive (or negative) definite (or semidefinite), or indefinite.

$M_1 = \det(-3) = -3$ ← nonzero failure of pos. def conditions

$M_2 = \det\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} = 6$

$M_3 = \det\begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 2$ ← nonzero failure of neg. def. conditions

So Q is indefinite.

Alt.: $Q(\vec{e}_1) = -3 < 0$, $Q(\vec{e}_4) = 5 > 0$ $\Rightarrow$ Q is indefinite