EXAM 3

Math 102, Fall 2009-2010, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(15 pts)* Find all of the critical points of the function $f(x,y) = \frac{1}{3}x^3 - 4xy^2 + y^2$.

$$\nabla f = \begin{pmatrix} x^2 - 4y^2 \\ -8xy + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \pm 2y$$

Case 1: $x = 2y \Rightarrow -16y^2 + 2y = 0 \Rightarrow y = 0$ or $y = 1/8$

Critical points: $\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/8 \end{pmatrix}}$

Case 2: $x = -2y \Rightarrow 16y^2 + 2y = 0 \Rightarrow y = 0$ or $y = -1/8$

Critical points: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\boxed{\begin{pmatrix} 1/4 \\ -1/8 \end{pmatrix}}$

2. *(15 pts)* Find the absolute maximum value of the function $g(x,y) = 2xy - 16x^2 - 4y^2$ inside the interior of the regular octogon centered at the origin with vertices at the points $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}})$.

Region is open and convex;

for all (x,y)

$$H = \begin{pmatrix} -16 & 2 \\ 2 & -4 \end{pmatrix}$$ is negative definite because

$\det(-16) < 0$ and $\det\begin{pmatrix} -16 & 2 \\ 2 & -4 \end{pmatrix} > 0$

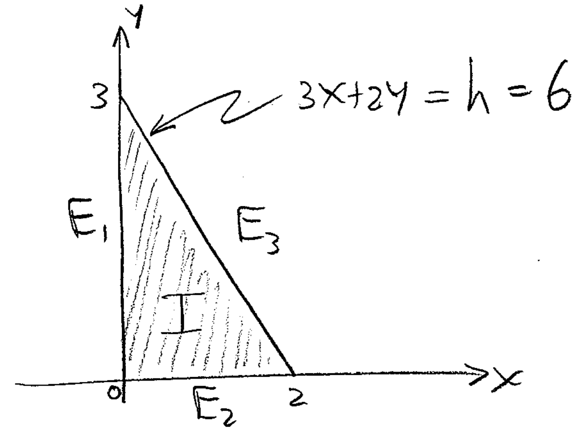
So any critical point must be an absolute max.

$$\nabla f = \begin{pmatrix} 2y - 32x \\ 2x - 8y \end{pmatrix} = \vec{0} \text{ only at } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the absolute maximizer, and

$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ is the absolute maximum.

3. *(15 pts)* Find the absolute maximum value of the function $f(x, y) = x^2y^3$ on the solid triangle with vertices at $(0, 0)$, $(2, 0)$, $(0, 3)$.



Note, $f=0$ on all 3 vertices and on E_1, E_2.

On interior I:

$$\nabla f = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x=0 \text{ or } y=0, \text{ neither possible in } I.$$

So no interior critical points.

On edge E_3:

$$\nabla h = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \neq \vec{0} \qquad \nabla f = \lambda \nabla h \Rightarrow \begin{pmatrix} 2xy^3 \\ 3x^2y^2 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \lambda = \tfrac{2}{3}xy^3 = \tfrac{3}{2}x^2y^2 \quad (\text{note } x \neq 0 \neq y \text{ on } E_3)$$

$$\Rightarrow x = \tfrac{4}{9}y$$

$$\Rightarrow 3\left(\tfrac{4}{9}y\right) + 2y = 6 \Rightarrow \tfrac{30}{9}y = 6$$

$$\Rightarrow y = \tfrac{54}{30} = \tfrac{9}{5}$$

$$x = \tfrac{4}{5}$$

Domain is closed & bounded, f is continuous, so abs. max exists.

$$f\begin{pmatrix} 4/5 \\ 9/5 \end{pmatrix} = \left(\tfrac{4}{5}\right)^2 \left(\tfrac{9}{5}\right)^3 = \frac{11664}{3125} > 0, \text{ so it is the}$$

absolute max.

4. *(20 pts)* Suppose we are trying to find the maximum value of the function $f : \mathbb{R}^4 \to \mathbb{R}$, subject to the three constraints $h_1 = 0$, $h_2 = 0$, $h_3 = 0$. All of these functions are known to be smooth. We also know the following about the points $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ on the constraint set:

$$\nabla f(\vec{a}_1) = (3,2,5,2) \quad \nabla h_1(\vec{a}_1) = (1,3,2,0) \quad \nabla h_2(\vec{a}_1) = (-2,1,4,0) \quad \nabla h_3(\vec{a}_1) = (4,0,1,0)$$

$$\nabla f(\vec{a}_2) = (3,1,1,2) \quad \nabla h_1(\vec{a}_2) = (1,0,0,2) \quad \nabla h_2(\vec{a}_2) = (0,1,0,0) \quad \nabla h_3(\vec{a}_2) = (2,0,1,0)$$

$$\nabla f(\vec{a}_3) = (0,2,5,6) \quad \nabla h_1(\vec{a}_3) = (1,2,0,0) \quad \nabla h_2(\vec{a}_3) = (5,1,0,0) \quad \nabla h_3(\vec{a}_3) = (3,4,0,0)$$

Which of these points are critical points? (Make sure to justify your answers!)

At $\vec{a}_1$: $\nabla f \neq \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3$, b/c last coordinates are all 0 on R.H.S.

$$\text{rank}\begin{pmatrix} \nabla h_1 \\ \nabla h_2 \\ \nabla h_3 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & 3 & 2 & 0 \\ -2 & 1 & 4 & 0 \\ 4 & 0 & 1 & 0 \end{pmatrix} \text{ is } 3$$

because $\det\begin{pmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ 4 & 0 & 1 \end{pmatrix} = 4(10) + 1(7) \neq 0$

So $\vec{a}_1$ is not a critical point.

At $\vec{a}_2$: $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \to \lambda_2 = 1, \; \lambda_3 = 1, \; \lambda_1 = 1$ and these values do work for 1st coord.

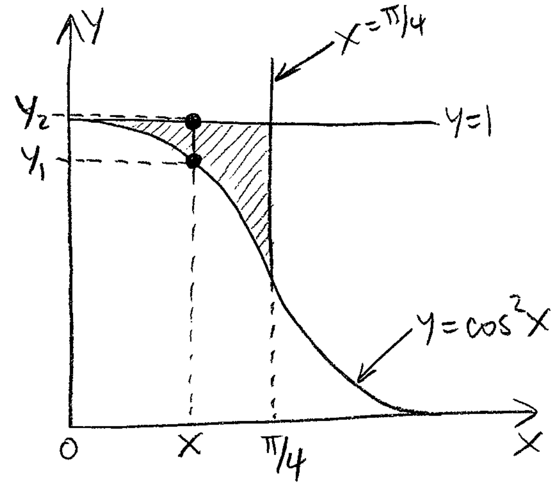
So $\vec{a}_2$ is a Lagrange critical point.

At $\vec{a}_3$: $\nabla f \neq \lambda_1 \nabla h_1 + \lambda_2 \nabla h_2 + \lambda_3 \nabla h_3$ b/c of 0's in 3rd, 4th coords.

$$\text{rank}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{pmatrix} \leq 2 < 3$$, b/c of zero columns,

So $\vec{a}_3$ is a degenerate critical point

5. *(15 pts)* Compute the double integral of the function $f(x, y) = \sec^2 x$ on the region in the first quadrant with $y \le 1$, $x \le \pi/4$, and $y \ge \cos^2 x$.



$x \in [0, \pi/4]$

For a fixed x,

$y \in [y_1, y_2]$

(x, y_1) on $y = \cos^2 x$

$\Rightarrow y_1 = \cos^2 x$

(x, y_2) on $y = 1$

$\Rightarrow y_2 = 1$

So $\iint f \, dA = \int_0^{\pi/4} \int_{\cos^2 x}^{1} (\sec^2 x) \, dy \, dx$

$$= \int_0^{\pi/4} \left(y \sec^2 x \Big]_{y=\cos^2 x}^{y=1} \right) dx$$

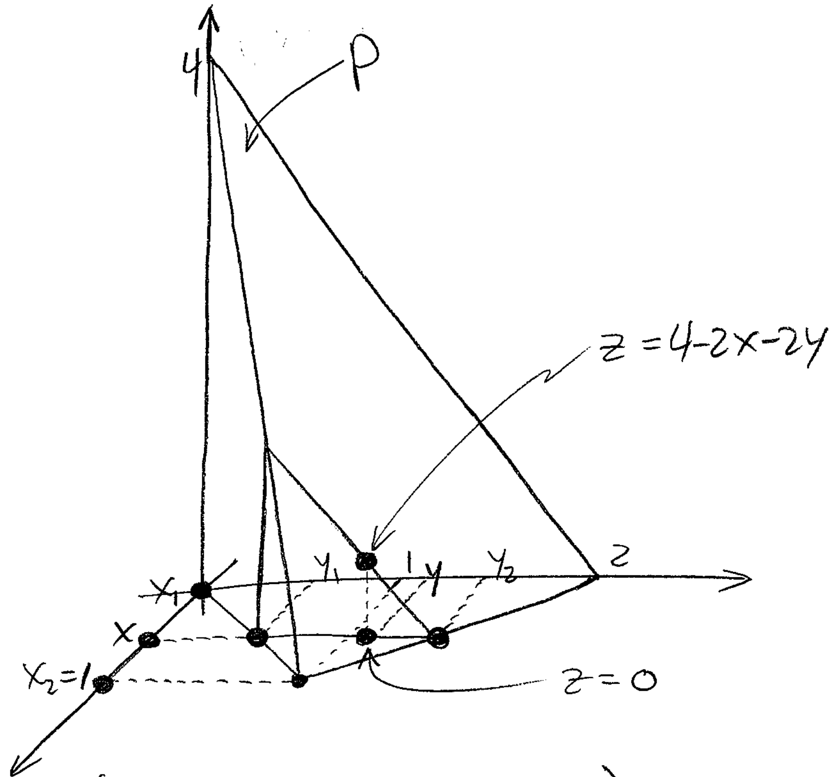
$$= \int_0^{\pi/4} \sec^2 x - 1 \, dx$$

$$= \tan x - x \Big]_0^{\pi/4}$$

$$= \boxed{1 - \frac{\pi}{4}}$$

6. *(20 pts)* Set up, but do not evaluate, a triple nested integral representing the mass of the solid tetrahedron with vertices at $(0,0,0)$, $(1,1,0)$, $(0,2,0)$, and $(0,0,4)$, where the density is given by $\delta(x,y,z) = x^2yz^3$.

$P \parallel \begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-2\\4\end{pmatrix}$



$\vec{n} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\-2\\4\end{pmatrix} = \begin{pmatrix}4\\4\\2\end{pmatrix}$

Eq. of P is $\vec{n}\cdot\vec{x} = \vec{n}\cdot\begin{pmatrix}0\\2\\0\end{pmatrix}$

$4x+4y+2z = 8$

$2x+2y+z = 4$

① $x \in [0,1]$; ② $y \in [y_1, y_2]$ $\quad (y_1 = x \; ; \; y_2 = 2-x)$

③ $z \in [0, 4-2x-2y]$

So $m = \iiint dm = \iiint \delta \, dV$

$$= \int_0^1 \int_x^{2-x} \int_0^{4-2x-2y} (x^2yz^3) \, dz \, dy \, dx$$