EXAM 1

Math 102, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

"I have adhered to the Duke Community Standard in completing this examination."

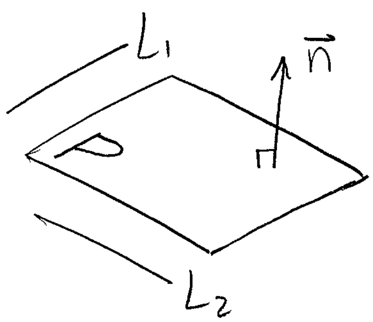
Signature: ________________________

Total Score ____________ (/100 points)

1. *(15 pts)* Find the equation of the unique plane that contains the point $(5, 5, 2)$ and is parallel to the lines parametrized by $(3-4t, 2t+1, 4t)$ and $(t, 2t-1, 3-t)$.

$$L_1: \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

$$L_2: \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$



Cross prod: $\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ -10 \end{pmatrix}$

So we can choose $\vec{n} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\perp$ to both lines and thus also the plane.

Eq. of plane is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$$

$$\boxed{x + z = 7}$$

2. *(15 pts)* Find the inverse of the matrix A defined by

$$A = \begin{pmatrix} 3 & -4 & 7 \\ 0 & 0 & -1 \\ -2 & 3 & 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 11 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} (1) \\ (2) \\ (3)+(1) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 11 & 1 & 0 & 1 \\ 3 & -4 & 7 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (3) \\ (1) \\ (2) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 11 & 1 & 0 & 1 \\ 0 & -1 & -26 & -2 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1) \\ (2)-3(1) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 37 & 3 & 0 & 4 \\ 0 & -1 & -26 & -2 & 0 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1)-(2) \\ (2) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 37 & 4 \\ 0 & -1 & 0 & -2 & -26 & -3 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} (1)+37(3) \\ (2)-26(3) \\ (3) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 37 & 4 \\ 0 & 1 & 0 & 2 & 26 & 3 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} (1) \\ -(2) \\ -(3) \end{array}$$

$$A^{-1} = \begin{pmatrix} 3 & 37 & 4 \\ 2 & 26 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

3. *(15 pts)* The system $A\vec{x} = \vec{b}$ relates the variables $x_1, \ldots, x_6$. The reduced row echelon form of A is the matrix below, and the variables b_1, b_2, b_3 are exogenous for this problem.

$$\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 6 & 4 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Which of the following sets of variables could be interpreted as endogenous in this system? (Make sure to justify each of your answers.)

(a) $\{x_1, x_3, x_4\}$

(b) $\{x_3, x_4, x_6\}$

(c) $\{x_4, x_5, x_6\}$

(a) Columns 1, 3, 4 of the above make $\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. This is not nonsingular, so this set of variables cannot be viewed as endogenous.

(b) Columns 3, 4, 6 make $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The determinant of this matrix is $-6 \neq 0$, so it is nonsingular, so this set can be viewed as endogenous.

(c) Columns 4, 5, 6 make $\begin{pmatrix} 6 & 4 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The determinant of this matrix is 0. So, it is not nonsingular, so this set cannot be viewed as endogenous.

4. *(15 pts)* Use pivots to show that it is not possible to have a linearly independent collection of 4 vectors in $\mathbb{R}^3$. (Make sure to explain all of the steps in your reasoning.)

$\{\vec{v}_1, \dots, \vec{v}_4\}$ l.i. $\Longleftrightarrow$ $c_1\vec{v}_1 + \dots + c_4\vec{v}_4 = \vec{0}$ has unique sols.

$\Longleftrightarrow$ $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_4 \\ | & & | \end{pmatrix}$ has uniqueness

$\Longleftrightarrow$ rref(A) has a pivot in every column

$\Longleftrightarrow$ rank (A) = 4

But A has only three rows, so rank (A) ≤ 3.

So these vectors cannot be l.i.

5. *(10 pts)* Compute the determinant of the matrix M given below.

$$M = \begin{pmatrix} 4 & 1 & -1 \\ 3 & 2 & 2 \\ 7 & 5 & 0 \end{pmatrix}$$

Expanding along 3rd column:

$\det A = (+)(-1)\det\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} + (-)(2)\det\begin{pmatrix} 4 & 1 \\ 7 & 5 \end{pmatrix} + (+)(0)\det\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$= (-1) + (-26) + (0)$

$= -27$

6. *(15 pts)* Determine if the collection $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of vectors spans $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We need every $\vec{b} \in \mathbb{R}^3$ to be written as

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$$

So we need

$$A\vec{c} = \vec{b}, \quad A = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 0 & 0 \\ 4 & -3 & 1 \end{pmatrix}$$

to have existence for all $\vec{b}$.

So we need A to be nonsingular, and $\det(A) \neq 0$.

We compute the determinant along 2nd row:

$$\det A = (-)(-2)\det\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} + 0 + 0$$

$$= 10 \neq 0$$

So A is nonsingular, has existence, and so these vectors <u>do</u> span $\mathbb{R}^3$.

7. *(15 pts)* Write the matrix A below as a product of elementary matrices.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix} \begin{array}{l} ① \\ ② - 2① \end{array} \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{array}{l} ① \\ ②/-4 \end{array} \qquad E_3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{array}{l} ① - 3② \\ ② \end{array}$$

This row reduction is written as

$$E_3 E_2 E_1 A = I$$

So

$$A = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}}$$