EXAM 2

Math 102, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number ________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. ________

2. ________

3. ________

4. ________

5. ________

6. ________

Total Score ________ (/100 points)

1. *(15 pts)* Find a basis for the span of the vectors below.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 4 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -6 & -2 \\ 0 & 0 & -2 \\ 0 & -3 & 0 \end{pmatrix} \begin{array}{l} ① \\ ② - 3① \\ ③ - 2① \\ ④ - 2① \end{array}$$

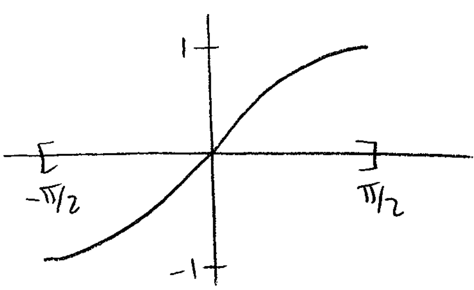
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{array}{l} ① \\ ④/-3 \\ ③/-2 \\ ②/-2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} ① - 2② \\ ② \\ ③ \\ ④ - 3② \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} ① - ③ \\ ② \\ ③ \\ ④ - ③ \end{array}$$

There is a pivot in every column so these vectors are independent.
So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis.

2. *(15 pts)* The function $f : \mathbb{R}^1 \to \mathbb{R}^1$ computed by $f(x) = \sin(x)$ is not invertible. Find a way to restrict the domain and the target to form a new function $g : A \to B$ computed by $g(x) = \sin(x)$ that is invertible.



$g : [-\pi/2, \pi/2] \to [-1, 1]$

is 1-1 and onto, and

thus invertible.

3. *(15 pts)* Consider the function $f(x,y,z) = xze^{x+yz+2z}$. Use the total derivative to estimate the value of $f(2.01, .02, -.99)$.

$\vec{x} = (2.01, .02, -.99)$ $\vec{a} = (2, 0, -1)$ $d\vec{x} = (.01, .02, .01)$

$f_x = z e^{x+yz+2z} + xze^{x+yz+2z}(1)$

$f_y = xze^{x+yz+2z}(z)$

$f_z = xe^{x+yz+2z} + xze^{x+yz+2z}(y+2)$

$$\left.\begin{aligned} f_x(\vec{a}) &= -1 - 2 = -3 \\ f_y(\vec{a}) &= 2 \\ f_z(\vec{a}) &= 2 + (-4) = -2 \end{aligned}\right\} \Rightarrow Df = \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix}$$

$f(\vec{a}) = -2$

$$f(\vec{x}) \approx f(\vec{a}) + Df \cdot d\vec{x}$$

$$= -2 + \begin{pmatrix} -3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} .01 \\ .02 \\ .01 \end{pmatrix}$$

$$= -2 - .01 = \boxed{-2.01}$$

4. *(20 pts)* We have $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$. Suppose that

$$\frac{\partial f}{\partial x}(3,3) = 4 \quad \text{and} \quad \frac{\partial f}{\partial y}(3,3) = 2$$

Compute $\frac{\partial z}{\partial \theta}$ at a point (r, θ) where $x = 3$ and $y = 3$.

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= (4)(-r\sin\theta) + (2)(r\cos\theta)$$

$$= -4r\sin\theta + 2r\cos\theta$$

We can use $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving $x=3, y=3$

$$\frac{\partial z}{\partial \theta} = -4(3\sqrt{2})(\sqrt{1/2}) + 2(3\sqrt{2})(\sqrt{1/2})$$

$$= -6$$

(expressed as a unit vector)

5. *(15 pts)* In what direction from the point $(2,3,1)$ is the function $f(x,y,z) = x^2y + yz^3$ increasing the fastest, and what is the directional derivative in that direction?

$$\nabla f = (2xy,\ x^2 + z^3,\ 3yz^2)$$

$$\nabla f(2,3,1) = (12, 5, 9)$$

$$\|\nabla f\| = \sqrt{144 + 25 + 81} = \sqrt{250} = 5\sqrt{10}$$

$$\text{Dir. of fastest increase} = \frac{\nabla f}{\|\nabla f\|} = \frac{(12,5,9)}{5\sqrt{10}} = \vec{u}$$

$$D_{\vec{u}} f = \|\nabla f\| = 5\sqrt{10}$$

6. *(20 pts)* We consider here the surface S with equation $x^2 - y^2 = z$.

(a) Is S the graph of a function $f : \mathbb{R}^a \longrightarrow \mathbb{R}^b$? If so, find a, b, and the formula for the function f.

This is the graph $z = f(x,y)$ of the function

$f: \mathbb{R}^2 \to \mathbb{R}^1$ given by $f(x,y) = x^2 - y^2$

(b) Is S a level set of a function $g : \mathbb{R}^c \longrightarrow \mathbb{R}^d$? If so, find c, d, and the formula for such a function g.

This is the level set $g = 0$ of the function

$g: \mathbb{R}^3 \to \mathbb{R}^1$ given by

$$g(x,y,z) = x^2 - y^2 - z$$