EXAM 3

Math 102, 2009-2010 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number__________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: __________________

1. ________
2. ________
3. ________
4. ________
5. ________
6. ________

Total Score ________ (/100 points)

1. *(15 pts)* The variables w, x, y, and z are related by the equation $F(w, x, y, z) = 0$, where F is C^1. At the point (w^*, x^*, y^*, z^*) on this set, the direction of fastest increase of F is in the direction of the vector $(3, -2, 0, 4)$.

   Given this information, which of the variables can be concluded to be locally functions of the other three variables on this set? (Make sure to justify your answers!)

We are given that $\nabla F = k \begin{pmatrix} 3 \\ -2 \\ 0 \\ 4 \end{pmatrix}$, with $k > 0$

So: $\frac{\partial F}{\partial w} \neq 0$, $\frac{\partial F}{\partial x} \neq 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} \neq 0$

By the implicit function theorem then, we know w, x, and z each can be viewed locally as functions of the other three variables.

And, y can not be concluded to be locally a function of the other three variables.

2. *(20 pts)* The variables p, q, r, s, and t are related by the system

$$\begin{aligned} F_1 = \quad pqr^2 - 2rs^2t &= -1 \\ F_2 = p^2q^2r + 3r^2st^2 &= 4 \end{aligned}$$

Show that near the point $(p^*, q^*, r^*, s^*, t^*) = (1, 1, 1, 1, 1)$ we can view r and t as implicitly defined functions of the other variables, and compute $\frac{\partial r}{\partial p}$ at this point.

$$D_{rt}\vec{F} = \begin{pmatrix} 2pqr - 2s^2t & -2rs^2 \\ p^2q^2 + 6rst^2 & 6r^2st \end{pmatrix}$$

$$D_{rt}\vec{F}\Big|_{\vec{x}^*} = \begin{pmatrix} 0 & -2 \\ 7 & 6 \end{pmatrix} \leftarrow \det = 14$$

We have $\det(D_{rt}\vec{F}) = 14 \neq 0$ at this point. So, r and t <u>can</u> be viewed locally as functions of the other variables.

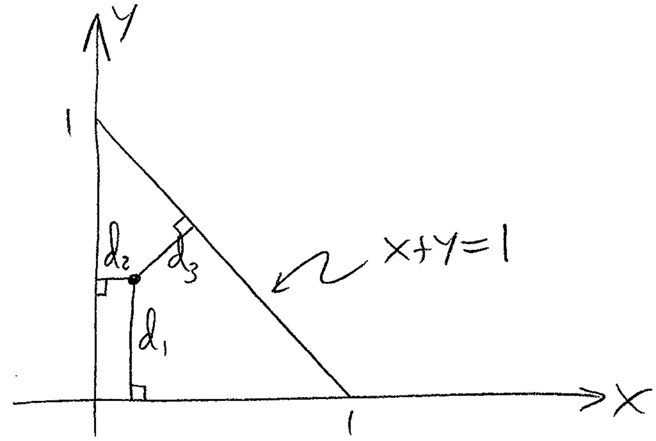
$$(D_{rt}\vec{F})\begin{pmatrix} \partial r/\partial p \\ \partial t/\partial p \end{pmatrix} + \begin{pmatrix} \partial F_1/\partial p \\ \partial F_2/\partial p \end{pmatrix} = \vec{0}$$

$$(D_{rt}\vec{F})\begin{pmatrix} \partial r/\partial p \\ \partial t/\partial p \end{pmatrix} + \begin{pmatrix} qr^2 \\ 2pq^2r \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 0 & -2 \\ 7 & 6 \end{pmatrix}\begin{pmatrix} \partial r/\partial p \\ \partial t/\partial p \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial r}{\partial p} = \frac{\det\begin{pmatrix} -1 & -2 \\ -2 & 6 \end{pmatrix}}{\det\begin{pmatrix} 0 & -2 \\ 7 & 6 \end{pmatrix}} = \frac{-10}{14} = \boxed{-\frac{5}{7}}$$

3. *(15 pts)* Let T be the solid triangle with vertices at the points $(1,0)$, $(0,1)$, $(0,0)$. Find the point on T that maximizes the product of the distances from the three edges. *(Hint: Recall that the distance from a point (x,y) to the line with equation $ax+by=c$ is $d = |ax+by-c|/\sqrt{a^2+b^2}$.)*



$$d_1 = y$$

$$d_2 = x$$

$$d_3 = \frac{1-x-y}{\sqrt{2}}$$

$$d_1 d_2 d_3 = \frac{xy(1-x-y)}{\sqrt{2}}$$

Equivalently, we can choose to maximize $f = xy(1-x-y)$.

First, note this is zero on all three edges and all three verts.

In the interior: $f = xy(1-x-y) = xy - x^2y - xy^2$

$$\nabla f = \begin{pmatrix} y - 2xy - y^2 \\ x - x^2 - 2xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} y(1-2x-y) = 0 \\ x(1-x-2y) = 0 \end{array}$$

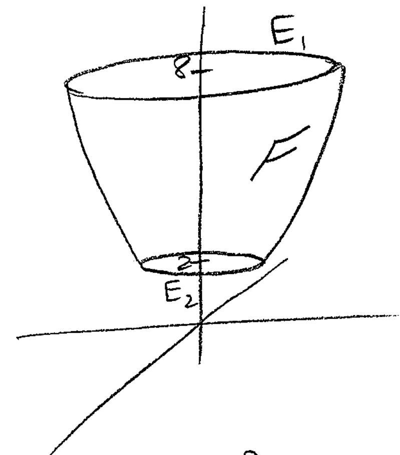
$$\Rightarrow \begin{array}{l} 2x+y = 1 \\ x+2y = 1 \end{array} \quad (\text{since } x,y \neq 0 \text{ on interior})$$

$$\Rightarrow \boxed{x = \tfrac{1}{3},\ y = \tfrac{1}{3}}$$

4. *(20 pts)* Find the point that maximizes the value of $f(x,y,z) = x + y - 3z$ subject to the constraints $z = x^2 + y^2$ and $2 \le z \le 8$.

$$h = x^2 + y^2 - z = 0 \qquad g_1 = z \le 8 \qquad g_2 = z \geq 2$$

$$Dh = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \qquad Dg_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad Dg_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



<u>On E_1:</u> 2 constraints; $h=0$, $g_1=8$

Deg: $Dh = \begin{pmatrix} 2x & 2y & -1 \\ 0 & 0 & 1 \end{pmatrix}$

rank $=2$ unless $x=y=0$, not on E_1

Lagr.: $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longleftrightarrow x = y \Rightarrow 2x^2 = 8$

$\Rightarrow x = \pm 2$

$\Rightarrow \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 8 \end{pmatrix}$ c.p.'s

<u>On E_2:</u> 2 constraints; $h=0$, $g_2=2$

Deg: $Dh = \begin{pmatrix} 2x & 2y & -1 \\ 0 & 0 & 1 \end{pmatrix}$

rank $=2$ unless $x=y=0$, not on E_2

Lagr.: $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Longleftrightarrow x = y \Rightarrow 2x^2 = 2$

$\Rightarrow x = \pm 1$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ c.p.'s

(over)

On F: 1 constraint: $h=0$

Deg: $\nabla h = \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \neq \vec{0}$

Lagr: $\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ -1 \end{pmatrix} \Rightarrow \lambda = 3$

$\Rightarrow x = y = \frac{1}{6}$

$\Rightarrow z = \frac{1}{18}$

This is not a c.p. because $z = \frac{1}{18} \ngeq 2$

We have four critical points in all:

$f\begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} = -20$

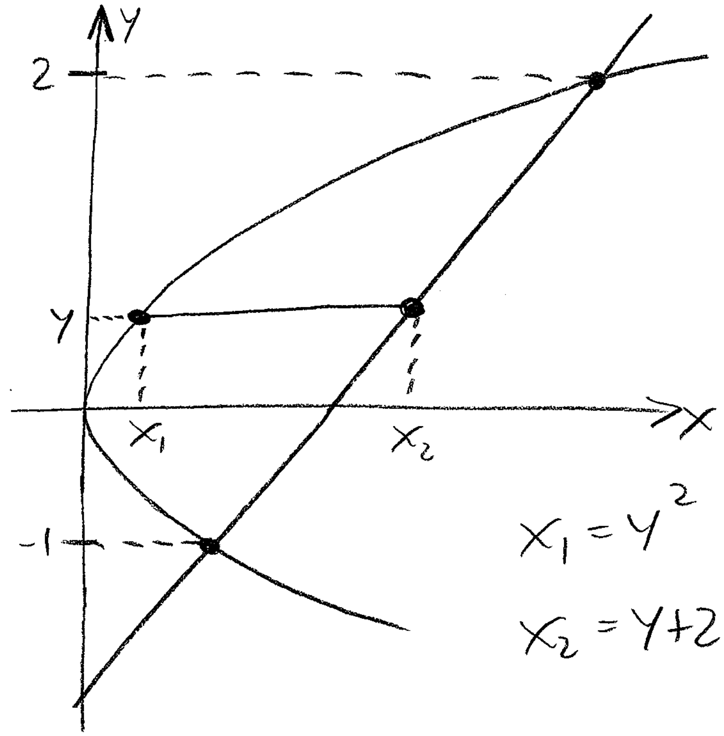
$f\begin{pmatrix} -2 \\ -2 \\ 8 \end{pmatrix} = -28$

$f\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = -4 \leftarrow$ Max at $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$f\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = -8$

5. *(15 pts)* Compute the double integral of the function $f(x,y) = xy$ over the region in the xy-plane bounded by the curves $x = y^2$ and $y = x - 2$.

$$\left.\begin{array}{l} x = y^2 \\ y = x-2 \end{array}\right\} \Longrightarrow \begin{array}{l} y^2 - y - 2 = 0 \\ (y+1)(y-2) = 0 \\ y = -1, 2 \end{array}$$



$$\iint f\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} (xy)\,dx\,dy$$

$$= \int_{-1}^{2} \left(\frac{1}{2}x^2 y\right]_{x=y^2}^{x=y+2} dy$$

$$= \frac{1}{2}\int_{-1}^{2} (y+2)^2 y - (y^2)^2 y \, dy$$

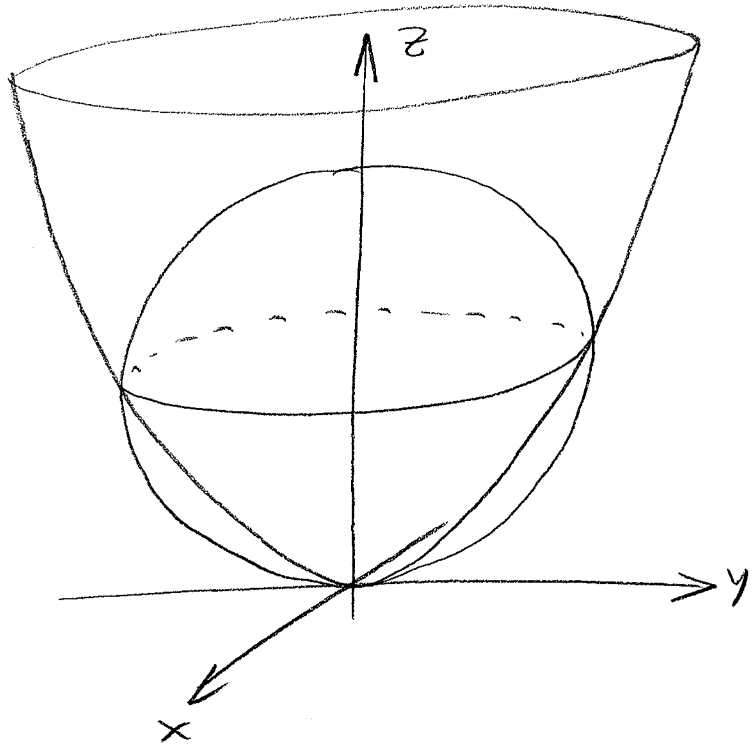
$$= \frac{1}{2}\int_{-1}^{2} y^3 + 4y^2 + 4y - y^5 \, dy$$

$$= \frac{1}{2}\left(\frac{1}{4}y^4 + \frac{4}{3}y^3 + 2y^2 - \frac{1}{6}y^6\right]_{-1}^{2}$$

$$= \frac{1}{2}\left[\left(4 + \frac{32}{3} + 8 - \frac{64}{6}\right) - \left(\frac{1}{4} - \frac{4}{3} + 2 - \frac{1}{6}\right)\right]$$

$$= \frac{1}{2}\left(12 - \frac{3}{4}\right) = \boxed{\frac{45}{8}}$$

6. *(15 pts)* Write a triple nested integral (but do not evaluate it!) that represents the mass of the region defined by $z \geq x^2 + y^2$ and $x^2 + y^2 + (z-1)^2 \leq 1$, where the density is given by the function $\delta(x, y, z) = e^{xyz}$.



Edge:

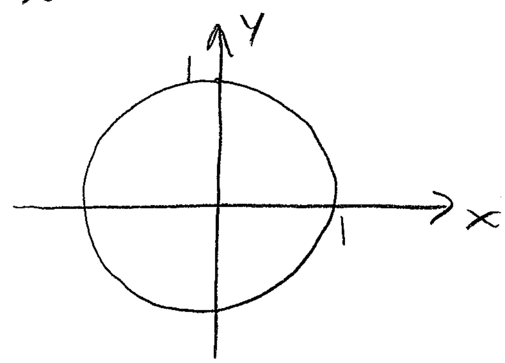
$$z = x^2 + y^2, \quad x^2 + y^2 + (z-1)^2 = 1$$

$$z + (z-1)^2 = 1$$

$$z^2 - z = 0$$

$$z = 1, \quad x^2 + y^2 = 1$$

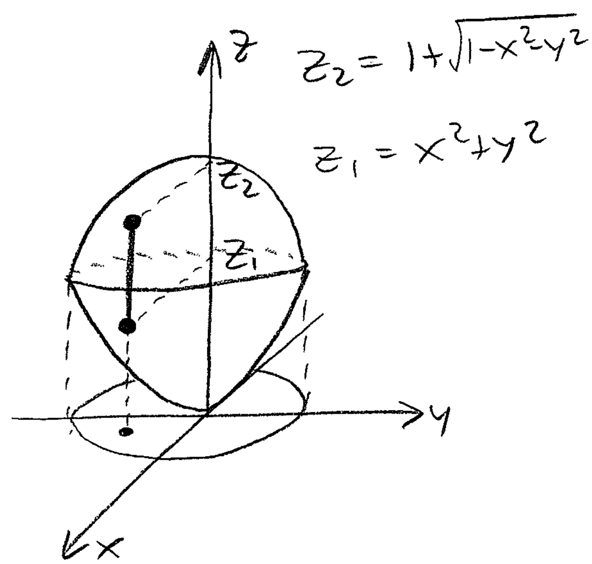
Proj. to xy-plane then is the unit disk.



x: $x \in [-1, 1]$

y: $y \in \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right]$

Above a point on the unit disk, z ranges from the paraboloid up to the sphere.



So

$$\iiint \delta \, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1+\sqrt{1-x^2-y^2}} e^{xyz} \, dz \, dy \, dx$$