

EXAM 1

Math 102, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number _____

1. _____

"I have adhered to the Duke Community
Standard in completing this
examination."

2. _____

3. _____

Signature: _____

4. _____

5. _____

6. _____

7. _____

8. _____

Total Score _____ (/100 points)

1. (10 pts) Suppose that the matrix A is as below, and the matrix B has three columns, with the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Show that the columns of the product matrix $C = BA$ must be linearly dependent.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix}$$

$$C = \begin{pmatrix} \downarrow & \downarrow & \downarrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 5 \end{pmatrix}$$

has column vectors

$$\vec{w}_1 = 1\vec{v}_1 + 1\vec{v}_2 + 2\vec{v}_3$$

$$\vec{w}_2 = 2\vec{v}_1 + 1\vec{v}_2 + 3\vec{v}_3$$

$$\vec{w}_3 = 3\vec{v}_1 + 2\vec{v}_2 + 5\vec{v}_3$$

This means that $\vec{w}_3 = \vec{w}_1 + \vec{w}_2$; so, these vectors are l.d.

2. (12 pts) A system of equations involving the variables x_1, x_2, x_3, x_4, x_5 is written as $A\vec{x} = \vec{b}$, where the matrix A is given below, along with its reduced row echelon form. Which of the following collections of vectors can be viewed as endogenous in this system?

$$A = \begin{pmatrix} 0 & -1 & -2 & -1 & 1 \\ -5 & 4 & -2 & -11 & -1 \\ 3 & -3 & 0 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) $\{x_1, x_2\}$ \leftarrow not enough variables - not an endogenous set.

(b) $\{x_2, x_4, x_5\}$

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \textcircled{2} \\ \textcircled{1}/3 \\ \textcircled{3} \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \textcircled{1} - \textcircled{2} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}$$

The matrix of these columns from A is nonsingular, so this is an endogenous set.

(c) $\{x_2, x_3, x_4\}$

$$\begin{pmatrix} 0 & 2 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The det of the matrix of these columns from A is 0, so the matrix is not nonsingular, and thus this is not an endogenous set.

3. (5 pts) Given the matrix M below, for what vectors $\vec{b} = (b_1, b_2, b_3)$ does the system $M\vec{x} = \vec{b}$ have solutions? In the cases where those solutions exist, find the complete set of those solutions.

$$M = \begin{pmatrix} 1 & -2 & 1 & -4 \\ 2 & -1 & 8 & 1 \\ 3 & -5 & 5 & -9 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 2 & -1 & 8 & 1 & b_2 \\ 3 & -5 & 5 & -9 & b_3 \end{array} \right) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 0 & 3 & 6 & 9 & b_2 - 2b_1 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \end{array} \right) \begin{array}{l} \textcircled{1} \\ \textcircled{2} -2\textcircled{1} \\ \textcircled{3} -3\textcircled{1} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -4 & b_1 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \\ 0 & 3 & 6 & 9 & b_2 - 2b_1 \end{array} \right) \begin{array}{l} \textcircled{1} \\ \textcircled{3} \\ \textcircled{2} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 2 & -5b_1 + 2b_3 \\ 0 & 1 & 2 & 3 & b_3 - 3b_1 \\ 0 & 0 & 0 & 0 & 7b_1 + b_2 - 3b_3 \end{array} \right) \begin{array}{l} \textcircled{1} + 2\textcircled{2} \\ \textcircled{2} \\ \textcircled{3} - 3\textcircled{2} \end{array}$$

need

$$7b_1 + b_2 - 3b_3 = 0$$

$$x_1 = -5b_1 + 2b_3 - 5x_3 - 2x_4$$

$$x_2 = -3b_1 + b_3 - 2x_3 - 3x_4$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5b_1 + 2b_3 - 5x_3 - 2x_4 \\ -3b_1 + b_3 - 2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

4. (13 pts) Find the equation of the unique plane P that contains the lines parametrized by

$$\left\{ \begin{bmatrix} 3-2t \\ 4+2t \\ 1-t \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 2s+3 \\ 5s+4 \\ s+1 \end{bmatrix} \right\}$$

$$\left. \begin{array}{l} \vec{v}_1 = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \\ \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \end{array} \right\} \quad \vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} 7 \\ 0 \\ -14 \end{pmatrix}$$

Both lines go through $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

So plane has eq: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \mathbf{x}_0$

$$\boxed{7x - 14z = 7}$$

5. (10 pts) Determine if the angle between the the two vectors $(1, -3, 2)$ and $(10, 5, 3)$ is acute, obtuse, or right.

$$\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 5 \\ 3 \end{pmatrix} = 10 - 15 + 6 = 1$$

$$\|\vec{v}\| \|\vec{w}\| \cos \theta = 1 > 0$$

So $\theta < \frac{\pi}{2}$ and thus θ is acute.

6. (20 pts) Compute the inverse of the matrix A below, and compute the determinant of A .

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 8 & 7 \\ -4 & 11 & 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -3 & 8 & 7 & 0 & 1 & 0 \\ -4 & 11 & 11 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 3 & 1 & 0 \\ 0 & 3 & 7 & 4 & 0 & 1 \end{array} \right) \begin{array}{l} \textcircled{1} \\ \textcircled{2} + 3\textcircled{1} \\ \textcircled{3} + 4\textcircled{1} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3/2 & 1/2 & 0 \\ 0 & 3 & 7 & 4 & 0 & 1 \end{array} \right) \begin{array}{l} \textcircled{1} \\ \textcircled{2}/2 \\ \textcircled{3} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 1 & 0 \\ 0 & 1 & 2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & -3/2 & 1 \end{array} \right) \begin{array}{l} \textcircled{1} + 2\textcircled{2} \\ \textcircled{2} \\ \textcircled{3} - 3\textcircled{2} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11/2 & 11/2 & -3 \\ 0 & 1 & 0 & 5/2 & 7/2 & -2 \\ 0 & 0 & 1 & -1/2 & -3/2 & 1 \end{array} \right) \begin{array}{l} \textcircled{1} - 3\textcircled{3} \\ \textcircled{2} - 2\textcircled{3} \\ \textcircled{3} \end{array}$$

$\text{rref}(A) = I$, so

$$A^{-1} = \begin{pmatrix} 11/2 & 11/2 & -3 \\ 5/2 & 7/2 & -2 \\ -1/2 & -3/2 & 1 \end{pmatrix}$$

$\frac{1}{2} \det A = 1$, so

$$\det A = 2$$

7. (10 pts) The graph of the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined by $f(x, y) = x^2 - y^3$ is a level set of a function $g: \mathbb{R}^a \rightarrow \mathbb{R}^b$. Find a , b , and an explicit formula for such a function g .

Graph of f has equation $z = x^2 - y^3$

This is equivalent to $x^2 - y^3 - z = 0$, which is the equation for the zero level set of

$$g: \mathbb{R}^3 \rightarrow \mathbb{R}^1, \quad g(x, y, z) = x^2 - y^3 - z$$

8. (10 pts) The curve parametrized by $\vec{x}(t) = (x(t), y(t)) = (t^3, t^6 - 5)$ is the graph of a function $h: \mathbb{R}^c \rightarrow \mathbb{R}^d$. Find c , d , and an explicit formula for this function h . (Hint: Find an algebraic relationship between x and y and relate this to the graph construction.)

Every point on the parametrized curve satisfies the equation $y = x^2 - 5$. This is the equation for the graph of

$$h: \mathbb{R}^1 \rightarrow \mathbb{R}^1, \quad h(x) = x^2 - 5$$