

EXAM 2

Math 102, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number _____

1. _____

"I have adhered to the Duke Community
Standard in completing this
examination."

2. _____

3. _____

Signature: _____

4. _____

5. _____

6. _____

7. _____

8. _____

Total Score _____ (/100 points)

1. (12 pts) Use the total derivative to estimate the value of the function $f(x, y, z) = e^{xz} \ln(xy)$ at the point $\vec{x} = (1.01, e - .01, .98)$. (Leave your answer as an expression involving e .)

We choose $\vec{a} = (1, e, 1)$, $d\vec{x} = \vec{x} - \vec{a} = (.01, -.01, -.02)$

$$\frac{\partial f}{\partial x} = (ze^{xz})(\ln(xy)) + (e^{xz})\left(\frac{1}{x}\right)$$

$$\frac{\partial f}{\partial x}(\vec{a}) = 2e$$

$$\frac{\partial f}{\partial y} = (e^{xz})\left(\frac{1}{y}\right)$$

$$\frac{\partial f}{\partial y}(\vec{a}) = 1$$

$$\frac{\partial f}{\partial z} = (xe^{xz})(\ln(xy))$$

$$\frac{\partial f}{\partial z}(\vec{a}) = e$$

The total derivative is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \nabla f \cdot d\vec{x}$$

$$= \begin{pmatrix} 2e \\ 1 \\ e \end{pmatrix} \cdot \begin{pmatrix} .01 \\ -.01 \\ -.02 \end{pmatrix} = -.01$$

The estimate of $f(\vec{x})$ then is

$$f(\vec{x}) \approx f(\vec{a}) + df$$

$$\approx e + (-.01)$$

$$\approx \boxed{e - .01}$$

2. (13 pts) The "spherical coordinate system" involves variables ρ , ϕ , θ , which are related to the more familiar rectangular coordinates x , y , z by the equations

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

Suppose that $f : \mathbb{R}^3 \rightarrow \mathbb{R}^1$ is a differentiable function of x , y , and z . Find an expression for $\frac{\partial f}{\partial \phi}$ in terms of the spherical variables and the rectangular partial derivatives of f .

$$\begin{array}{ccc} \rho & & x \\ \phi & \rightarrow & y \\ \theta & & z \end{array} \rightarrow f$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \phi}$$

$$= \left(\frac{\partial f}{\partial x} \right) (\rho \cos \phi \cos \theta) + \left(\frac{\partial f}{\partial y} \right) (\rho \cos \phi \sin \theta) + \left(\frac{\partial f}{\partial z} \right) (-\rho \sin \phi)$$

3. (10 pts) Bob is standing on the side of a mountain (which you can view for this problem as being the graph of a differentiable function f), and you are down in the valley below talking to him through radios. Because of bad reception you are able to interpret only the following of his statements about the ground immediately surrounding the point where he is currently standing:

- (a) Looking directly east, the slope is 0.5.
- (b) The uphill direction is east of north.
- (c) The slope of the mountain in the uphill direction is 1.3.

Given this information, determine exactly what direction is straight uphill where Bob is currently standing.

(a) means that $\frac{\partial f}{\partial x} = 0.5$

(c) means that $\|\nabla f\| = 1.3$

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (1.3)^2$$

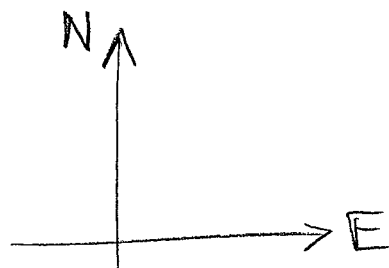
$$(0.5)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = (1.3)^2$$

$$\left(\frac{\partial f}{\partial y}\right)^2 = 1.44 = (1.2)^2$$

(b) tells us that $\frac{\partial f}{\partial y} > 0$, so $\frac{\partial f}{\partial y} = 1.2$

So $\nabla f = \begin{pmatrix} 0.5 \\ 1.2 \end{pmatrix}$, and thus the uphill

direction is $\frac{\nabla f}{\|\nabla f\|} = \begin{pmatrix} 5/13 \\ 12/13 \end{pmatrix}$



4. (15 pts) Consider the equation

$$3xe^xy^2z - x^2z^3 = 12e - 1$$

Near the point $(x^*, y^*, z^*) = (1, 2, 1)$, determine if x can be viewed as a function of y and z , and if so compute $\frac{\partial x}{\partial z}$.

$$\frac{\partial F}{\partial x} = (3x+3)e^xy^2z - 2xz^3, \text{ so } \frac{\partial F}{\partial x}(\vec{x}^*) = 12e - 2 \neq 0$$

So x is a fn of y and z locally. Take $\frac{\partial}{\partial z}$:

$$3\left(\frac{\partial x}{\partial z}\right)e^xy^2z + 3x\left(e^x\frac{\partial x}{\partial z}\right)y^2z + 3xe^x(y^2) - \left(2x\frac{\partial x}{\partial z}\right)z^3 - x^2(3z^2) = 0$$

$$\frac{\partial x}{\partial z} = \frac{3x^2z^2 - 3xe^xy^2}{3e^xy^2z + 3xe^xy^2z - 2xz^3}$$

$$\frac{\partial x}{\partial z}(1, 2, 1) = \frac{3-12e}{24e-2}$$

5. (10 pts) Show that the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y) = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}$$

is locally 1-1 and locally onto at every point in its domain.

$$Df = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$$\det(Df) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$$

So Df is non-singular, and thus both 1-1 and onto.

So f is locally 1-1 and locally onto.

6. (15 pts) Consider the system

$$F_1 = 3x^2y - xz + wz = 32$$

$$F_2 = xy - yz + 2wx = -4$$

Near the point $(w^*, x^*, y^*, z^*) = (1, 2, 3, 4)$, show that we can view y and z as functions of w and x , and compute $\frac{\partial z}{\partial w}$.

$$D_{yz} \vec{F} = \begin{pmatrix} 3x^2 & w-x \\ x-z & -y \end{pmatrix}, \quad D_{yz} \vec{F}(\vec{x}^*) = \begin{pmatrix} 12 & -1 \\ -2 & -3 \end{pmatrix}$$

At the point in question this has $\det = -38 \neq 0$.

So y, z can be viewed as fns of w, x .

$$(D_{yz} \vec{F}) \begin{pmatrix} \partial y / \partial w \\ \partial z / \partial w \end{pmatrix} + \begin{pmatrix} \partial F_1 / \partial w \\ \partial F_2 / \partial w \end{pmatrix} = \vec{0}$$

$$(D_{yz} \vec{F}) \begin{pmatrix} \partial y / \partial w \\ \partial z / \partial w \end{pmatrix} + \begin{pmatrix} z \\ 2x \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 12 & -1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} \partial y / \partial w \\ \partial z / \partial w \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$$

$$\frac{\partial z}{\partial w} = \frac{\det \begin{pmatrix} 12 & -4 \\ -2 & -4 \end{pmatrix}}{\det \begin{pmatrix} 12 & -1 \\ -2 & -3 \end{pmatrix}} = \frac{-56}{-38}$$

$$= \boxed{\frac{28}{19}}$$

7. (10 pts) Consider the quadratic form $Q(\vec{x}) = \vec{x} \cdot A\vec{x}$, where the symmetric matrix A is

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 3 & 2 & 1 & 0 \\ 3 & 4 & -6 & 2 & 5 & 6 & 4 \\ 4 & 3 & 2 & 0 & 4 & 7 & 8 \\ 5 & 2 & 5 & 4 & -3 & 2 & 12 \\ 6 & 1 & 6 & 7 & 2 & 4 & 1 \\ 7 & 0 & 4 & 8 & 12 & 1 & 10 \end{pmatrix}$$

Determine if this quadratic form is positive or negative definite or semidefinite, or indefinite. Make sure to explain your reasoning clearly.

By noting certain elements on the diagonal, we have

$$Q(\vec{e}_2) = 5 > 0$$

$$Q(\vec{e}_3) = -6 < 0$$

So Q is indefinite

8. (15 pts) Find all of the critical points of the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined by

$$f(x, y) = 2x^3 - 3x^2y + y^2 + y$$

$$\nabla f = \begin{pmatrix} 6x^2 - 6xy \\ -3x^2 + 2y + 1 \end{pmatrix} \Rightarrow x=0 \quad \underline{\text{or}} \quad x=y$$

$$\text{Case 1: } x=0 \Rightarrow 2y+1=0 \Rightarrow y = -\frac{1}{2}$$

$$\text{Case 2: } x=y \Rightarrow -3y^2 + 2y + 1 = 0 \Rightarrow y = \frac{-2 \pm \sqrt{4+12}}{-6}$$

$$y = 1 \quad \underline{\text{or}} \quad -\frac{1}{3}$$

So the critical pts are: $\begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}$