EXAM 3

Math 102, 2010-2011 Fall, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________
2. __________
3. __________
4. __________
5. __________

Total Score __________ (/100 points)

1. *(20 pts)* Find the absolute maximizer of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = -x^2 - 4y^2 + 3xy + 3x + 2y$$

$$\nabla f = \begin{pmatrix} -2x + 3y + 3 \\ 3x - 8y + 2 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -2 & 3 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -8 \end{pmatrix}^{-1} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \frac{\begin{pmatrix} -8 & -3 \\ -3 & -2 \end{pmatrix}}{7} \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 30/7 \\ 13/7 \end{pmatrix}$$

This is the only critical point.

$$H = \begin{pmatrix} -2 & 3 \\ 3 & -8 \end{pmatrix}$$

LPM's : $-2, 7$ ← same sign as $(-1)^k$

H is neg. def., thus neg. semidef, so the above critical point is the maximizer.

2. *(20 pts)* Find the maximizers and minimizers of the function f defined by $f(x, y, z) = x + 2y + 4z$ with the constraints that $y^2 + z^2 = 1$ and $2x - y + 3z = 2$.

$f = x + 2y + 4z \qquad h_1 = y^2 + z^2 \qquad h_2 = 2x - y + 3z$

$\nabla f = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \qquad Dh_1 = \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix} \qquad \nabla h_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Deg: Need rank $\begin{pmatrix} 0 & 2y & 2z \\ 2 & -1 & 3 \end{pmatrix} < 2$

$\Rightarrow y, z = 0 \leftarrow$ not possible given $y^2 + z^2 = 1$.

So, no deg. crit. pts.

Lagr.: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

1st coord says $1 = 2\lambda_2 \Rightarrow \lambda_2 = \frac{1}{2}$. Eq. becomes:

$\begin{pmatrix} 0 \\ 5/2 \\ 5/2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 2y \\ 2z \end{pmatrix}$

2nd, 3rd coords say $z = y$. Then

$1 = y^2 + z^2 = y^2 + (y)^2 = 2y^2$

$\Rightarrow y = \pm\sqrt{1/2} \Rightarrow z = \pm\sqrt{1/2}$

and $x = 1 + \frac{y}{2} - \frac{3z}{2}$

Lagr. crit. pts. are thus

$$\vec{x}_1 = \begin{pmatrix} 1-\sqrt{1/2} \\ \sqrt{1/2} \\ \sqrt{1/2} \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} 1+\sqrt{1/2} \\ -\sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix}$$

$$f(\vec{x}_1) = 1 + 5\sqrt{1/2} \qquad f(\vec{x}_2) = 1 - 5\sqrt{1/2}$$

greatest

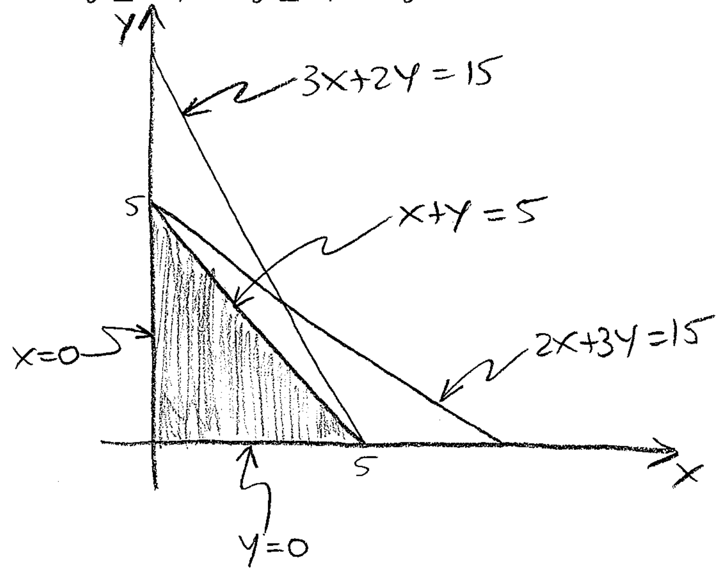
least

We know the optimizers exist because f is continuous and the domain (ellipse in $\mathbb{R}^3$) is closed & bounded.

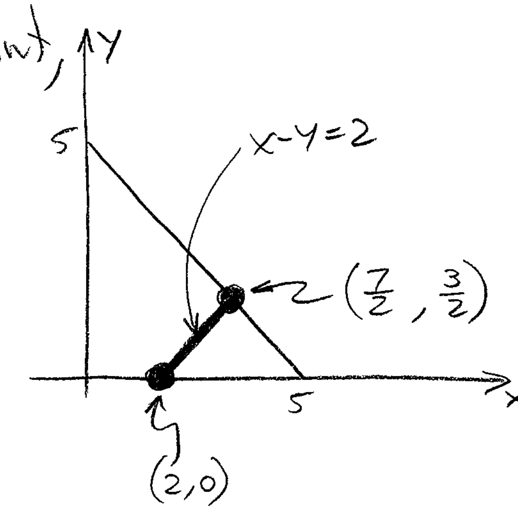
So $\vec{x}_1$ is the maximizer, and $\vec{x}_2$ is the minimizer.

3. *(20 pts)* Find the absolute maximizer of the function f defined by $f(x, y) = x^2 + y^2$ with constraints $x \geq 0$, $y \geq 0$, $2x + 3y \leq 15$, $3x + 2y \leq 15$, $x + y \leq 5$, $x - y = 2$.

The 3rd, 4th inequalities are unnecessary, as seen in the figure. The set defined by these inequalities is simply the triangle shown.



Combining with the equality constraint, the constraint set is the line segment shown.



The endpts $(2,0)$, $(\frac{7}{2}, \frac{3}{2})$ are critical points.

On the edge, we have one constraint: $h_1 = x - y = 2$

$$\nabla f = \lambda \nabla h_1 \qquad \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow x = -y$$

$$\Rightarrow x = 1,\ y = -1$$

(not on our constr. set, since $y \not\geq 0$).

So we have only 2 c.p.'s:

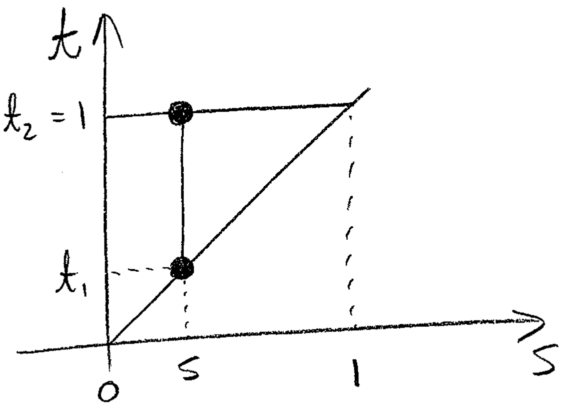
$f(2,0) = 4$

$f(\frac{7}{2}, \frac{3}{2}) = \frac{29}{2}$ ⇐ maximizer (since f cont., domain is closed, bdd.)

4. *(20 pts)* According to a market model that you have devised, the probability that the price of corn will rise above $7\frac{3}{8}$ in the next year is expressed by the integral of the function $f(s,t) = se^{s-t}$ on the domain defined by $s - t \leq 0$, $s \geq 0$, $t \leq 1$. Compute this probability.

$s \in [0,1]$

$t \in [s,1]$



$$\iint_D se^{s-t}\,dA = \int_0^1 \int_s^1 se^{s-t}\,dt\,ds$$

$$= \int_0^1 \left(-se^{s-t} \Big]_{t=s}^{t=1} \right) ds = \int_0^1 \left(s - se^{s-1} \right) ds$$

$$= \int_0^1 s\,ds - \int_0^1 se^{s-1}\,ds$$

$f = s \qquad g' = e^{s-1}$

$f' = 1 \qquad g = e^{s-1}$

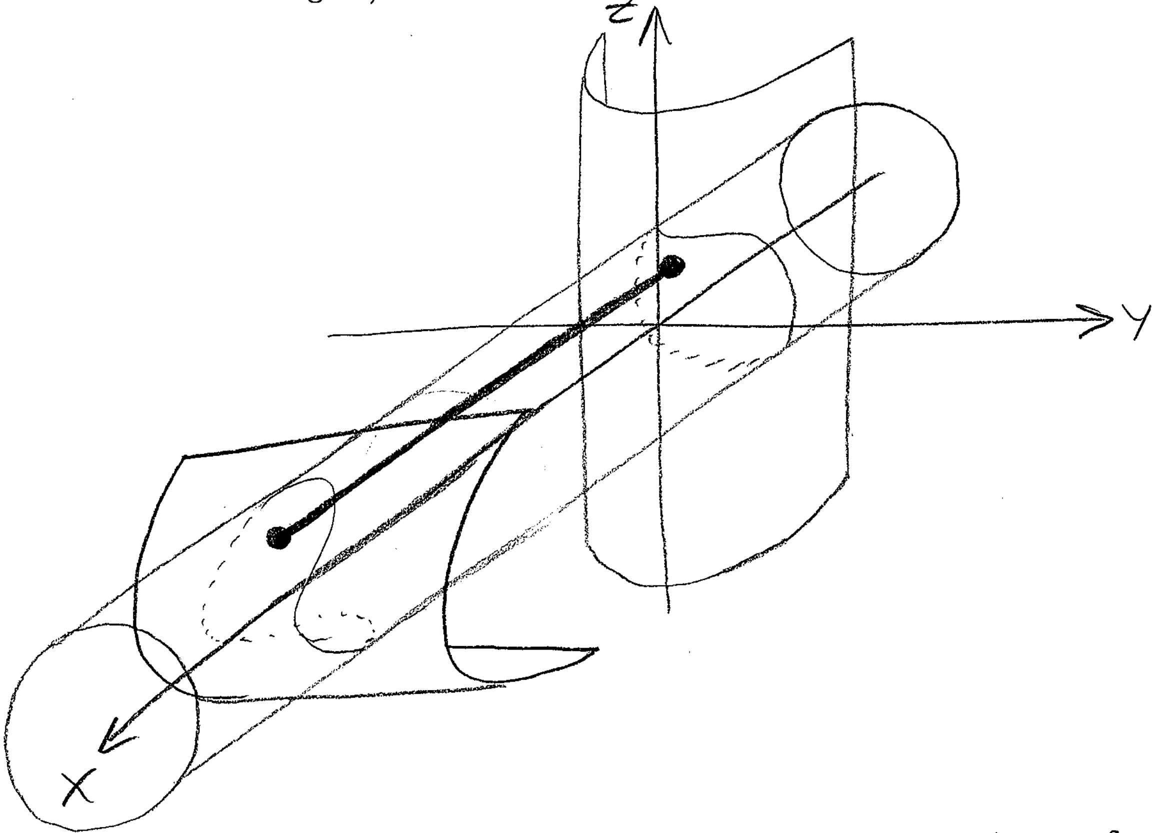
$$= \left(\tfrac{1}{2}s^2\right)\Big]_0^1 - \left(se^{s-1} - \int e^{s-1}\,ds \right)\Big]_0^1$$

$$= \frac{1}{2} - \left(se^{s-1} - e^{s-1} \right)\Big]_0^1$$

$$= \frac{1}{2} - \left((0) - \left(-\frac{1}{e}\right) \right)$$

$$= \boxed{\frac{1}{2} - \frac{1}{e}}$$

5. *(20 pts)* The domain D is bounded by the surfaces $y^2 + z^2 = 1$, $x = -y^2 - 3z^2$, and $x + y + z^2 = 5$. Write the triple integral $\iiint_D f\,dV$ as a triple nested integral. (Do not evaluate the integral.)



The projection to the yz-plane is the disk $y^2+z^2 \leq 1$

So, $y \in [-1,1]$, $z \in [-\sqrt{1-y^2}, \sqrt{1-y^2}]$.

The x bounds then are $x_1 = -y^2 - 3z^2$, $x_2 = 5 - y - z^2$

The nested integral then is

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-y^2-3z^2}^{5-y-z^2} (f)\, dx\, dz\, dy$$