EXAM 1

Math 102, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number________________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

Total Score ____________ (/100 points)

1. *(15 pts)* Find the complete set of solutions to the system of equations below.

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 &= 13 \\ 2x_1 + 4x_2 - x_3 + 6x_4 &= 19 \\ 11x_1 + 22x_2 - 7x_3 + 30x_4 &= 115 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 13 \\ 2 & 4 & -1 & 6 & 19 \\ 11 & 22 & -7 & 30 & 115 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 13 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 4 & 8 & -28 \end{array}\right) \begin{array}{l} (1) \\ (2) - 2(1) \\ (3) - 11(1) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 6 \\ 0 & 0 & 1 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} (1) + (2) \\ (2) \\ (3) - 4(2) \end{array}$$

$$x_1 = 6 - 2x_2 - 4x_4$$
$$x_3 = -7 \qquad - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 - 2x_2 - 4x_4 \\ x_2 \\ -7 - 2x_4 \\ x_4 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 0 \\ -7 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix}}$$

2. *(10 pts)* The rows of the 5x5 nonsingular matrices A and B have the following in common:

(a) B_1 is A_1 plus 2 times A_2.

(b) B_2 is 5 times A_5 minus 6 times A_3.

(c) B_3 is 8 times A_2 minus A_1.

(d) B_4 is 3 times A_5 minus A_4.

(e) B_5 is A_4.

Find the unique matrix C that satisfies $CA = B$.

$$\begin{pmatrix} C \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} = \begin{pmatrix} A_1 + 2A_2 \\ -6A_3 + 5A_5 \\ -A_1 + 8A_2 \\ -A_4 + 3A_5 \\ A_4 \end{pmatrix}$$

The rows of A are independent because A is nonsingular; so the expressions using them to describe the rows of B are unique.

We can find the rows of C by how each corresponding row of B is written as a linear combination of the rows of A.

So:

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 5 \\ -1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

3. *(15 pts)* Find the inverse matrix for the matrix N below.

$$N = \begin{pmatrix} 5 & 2 & 2 \\ 1 & -5 & 0 \\ -3 & 2 & -1 \end{pmatrix}$$

We row reduce $(N|I)$:

$$\left(\begin{array}{ccc|ccc} 5 & 2 & 2 & 1 & 0 & 0 \\ 1 & -5 & 0 & 0 & 1 & 0 \\ -3 & 2 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 5 & 2 & 2 & 1 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} ② \\ ① \\ ③ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 0 & 27 & 2 & 1 & -5 & 0 \\ 0 & -13 & -1 & 0 & 3 & 1 \end{array}\right) \begin{matrix} ① \\ ② - 5① \\ ③ + 3① \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -13 & -1 & 0 & 3 & 1 \end{array}\right) \begin{matrix} ① \\ ② + 2③ \\ ③ \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 10 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 13 & 16 & 27 \end{array}\right) \begin{matrix} ① + 5② \\ ② \\ ③ + 13② \end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 6 & 10 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -13 & -16 & -27 \end{array}\right) \begin{matrix} ① \\ ② \\ -③ \end{matrix}$$

this is I, so N is invertible

$$N^{-1} = \begin{pmatrix} 5 & 6 & 10 \\ 1 & 1 & 2 \\ -13 & -16 & -27 \end{pmatrix}$$

4. *(15 pts)* Suppose that the square matrix A is nonsingular. Show that A must be invertible.

If A is nonsingular, then $\mathrm{rref}(A) = I$. So we can represent its row reduction with elementary matrices by

$$E_k \cdots E_1 A = R = I$$

Writing $E_k \cdots E_1 = E$, this becomes

$$EA = I$$

So A is invertible, and its inverse is $A^{-1} = E$. ■

5. *(15 pts)* Compute the cosine of the angle between the vectors $\vec{a} = (1, 3, 2)$ and $\vec{b} = (4, -3, 1)$. Is this an acute or an obtuse angle?

$$\|\vec{a}\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

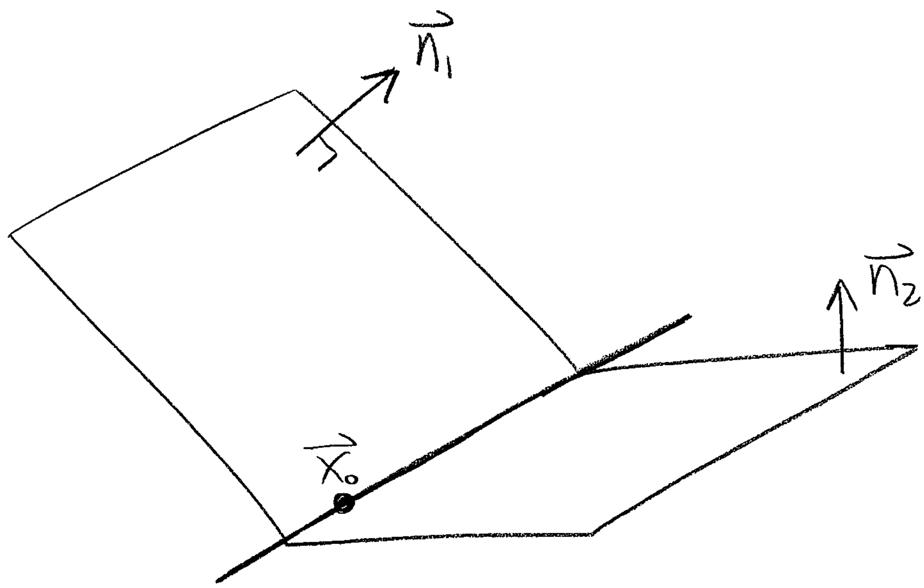
$$\|\vec{b}\| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26}$$

$$\vec{a} \cdot \vec{b} = (1)(4) + (3)(-3) + (2)(1) = -3$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{-3}{\sqrt{14}\sqrt{26}} = \boxed{\frac{-3}{2\sqrt{91}}}$$

The cosine is negative, so this angle is **obtuse.**

6. *(15 pts)* Find a parametrization of the line that is the intersection of the planes with equations $2x - 3y + 4z = 0$ and $x + y - 2z = 12$.



We solve the system

$$\begin{aligned} 2x - 3y + 4z &= 0 \\ x + y - 2z &= 12 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 12 \\ 2 & -3 & 4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 12 \\ 0 & -5 & 8 & -24 \end{array}\right) \begin{array}{l} ① \\ ② - 2① \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 12 \\ 0 & 1 & -8/5 & 24/5 \end{array}\right) \begin{array}{l} ① \\ ② / -5 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2/5 & 36/5 \\ 0 & 1 & -8/5 & 24/5 \end{array}\right) \begin{array}{l} ① - ② \\ ② \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 36/5 + \frac{2}{5} z \\ 24/5 + \frac{8}{5} z \\ z \end{pmatrix} = \boxed{\begin{pmatrix} 36/5 \\ 24/5 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2/5 \\ 8/5 \\ 1 \end{pmatrix}}$$

7. *(15 pts)* Suppose that the vectors $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ form a linearly independent set. Can you necessarily conclude that the vectors $\vec{p} = (a_1, b_1, c_1)$, $\vec{q} = (a_2, b_2, c_2)$, $\vec{r} = (a_3, b_3, c_3)$ also form a linearly independent set? If yes, explain how you know this must be true; if no, provide a counterexample.

$\{\vec{a}, \vec{b}, \vec{c}\}$ l.i. $\iff x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$ has ! sols

$$\iff \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \text{ has ! sols}$$

$$\iff \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ has } \det \neq 0$$

because $\det A = \det A^T$

$$\iff \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \text{ has } \det \neq 0$$

$$\iff \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0} \text{ has ! sols}$$

$\iff x_1\vec{p} + x_2\vec{q} + x_3\vec{r} = \vec{0}$ has ! sols

$\iff \{\vec{p}, \vec{q}, \vec{r}\}$ l.i.

■