

EXAM 2

Math 102, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number _____

"I have adhered to the Duke Community
Standard in completing this
examination."

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

Signature: _____

Total Score _____ (/100 points)

1. (pts) In this problem we consider the functions $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by the formulas

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy^2 \\ y^2 - x^2 \end{pmatrix} \quad \text{and} \quad g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ e^x \end{pmatrix}$$

Suppose we wish to compose these functions as $h = g \circ f$. Find the derivative matrix for this new function h .

To avoid confusion of variables, we rewrite as

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy^2 \\ y^2 - x^2 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad g \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv \\ e^u \end{pmatrix}$$

The derivative matrices are

$$Df = \begin{pmatrix} y^2 & 2xy \\ -2x & 2y \end{pmatrix} \quad Dg = \begin{pmatrix} v & u \\ e^u & 0 \end{pmatrix} \\ = \begin{pmatrix} y^2 - x^2 & xy^2 \\ e^{xy^2} & 0 \end{pmatrix}$$

The chain rule then gives us

$$Dh = D(g \circ f) = (Dg)(Df) \\ = \begin{pmatrix} y^2 - x^2 & xy^2 \\ e^{xy^2} & 0 \end{pmatrix} \begin{pmatrix} y^2 & 2xy \\ -2x & 2y \end{pmatrix} \\ = \begin{pmatrix} y^4 - 3x^2y^2 & 4xy^3 - 2x^3y \\ y^2e^{xy^2} & 2xye^{xy^2} \end{pmatrix}$$

2. (pts) It is known that the linear approximation to the function $P : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ at the point $(3, 2, 5)$ is $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$L(x, y, z) = (5 - 2x + 4y + z, -3 + x - 5y + 2z)$$

Find the value $P(3, 2, 5)$, and find the derivative matrix $DP(3, 2, 5)$.

$$\begin{aligned} L\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 1 \\ 1 & -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 1 \\ 1 & -5 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 1 \\ 1 & -5 & 2 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \\ z-5 \end{pmatrix} \\ &= \begin{pmatrix} 12 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 & 4 & 1 \\ 1 & -5 & 2 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \\ z-5 \end{pmatrix} \\ &= P\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + (DP(3, 2, 5)) (\vec{x} - (3, 2, 5)) \end{aligned}$$

$$\text{So } P(3, 2, 5) = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$DP(3, 2, 5) = \begin{pmatrix} -2 & 4 & 1 \\ 1 & -5 & 2 \end{pmatrix}$$

3. (pts) Find the equation for the tangent plane to the surface $x^2y - y^2z = xz^2 - 19$ at the point $(1, 3, 2)$.

Surface is a level set of $g = x^2y - y^2z - xz^2$, so

$$\nabla g|_{(1,3,2)} = \begin{pmatrix} 2xy - z^2 \\ x^2 - 2yz \\ -y^2 - 2xz \end{pmatrix} \Big|_{(1,3,2)} = \begin{pmatrix} 2 \\ -11 \\ -13 \end{pmatrix} \quad \text{is a normal vector to the tangent plane}$$

So the equation of the tangent plane is

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0 \quad \Rightarrow \quad 2x - 11y - 13z = -57$$

4. (pts) Compute the directional derivative of the function $f(x, y) = x^2y$ at the point $(1, 2)$ in the unit vector direction $(3, 4)/5$, directly from the definition (do not use the gradient vector!).

$$\vec{a} = (1, 2), \quad \vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$D_{\vec{u}} f(\vec{a}) = \left. \frac{d}{dt} \right|_{t=0} f(\vec{a} + t\vec{u})$$

$$= \left. \frac{d}{dt} \right|_{t=0} f\left(1 + \frac{3}{5}t, 2 + \frac{4}{5}t\right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left(1 + \frac{3}{5}t\right)^2 \left(2 + \frac{4}{5}t\right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left(1 + \frac{6}{5}t + \frac{9}{25}t^2\right) \left(2 + \frac{4}{5}t\right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left(2 + \frac{16}{5}t + \frac{42}{25}t^2 + \frac{36}{125}t^3\right)$$

$$= \left. \left(\frac{16}{5} + \frac{84}{25}t + \frac{108}{125}t^2\right) \right|_{t=0}$$

$$= \frac{16}{5}$$

5. (pts) The variables $a, b, c, d,$ and e are related by the system

$$\begin{aligned}a^2 - (b/c) + ade^2 &= -1 \\ abc + c^2d - e &= -2\end{aligned}$$

Show that, near the point $(a^*, b^*, c^*, d^*, e^*) = (0, 1, 1, 0, 2)$, we can view c and d as functions of $a, b,$ and $e,$ and compute their partial derivatives with respect to e at that point.

$$D_{cd} \vec{F} = \begin{pmatrix} \frac{\partial F_1}{\partial c} & \frac{\partial F_1}{\partial d} \\ \frac{\partial F_2}{\partial c} & \frac{\partial F_2}{\partial d} \end{pmatrix} = \begin{pmatrix} \frac{b}{c^2} & ae^2 \\ ab + 2cd & c^2 \end{pmatrix}$$

$$D_{cd} \vec{F}(\vec{x}^*) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det(D_{cd} \vec{F}(\vec{x}^*)) = 1 \neq 0,$ so c, d can be viewed as fns of a, b, e near this point.

To compute the partials, we have

$$(D_{cd} \vec{F}) \begin{pmatrix} \partial c / \partial e \\ \partial d / \partial e \end{pmatrix} + \frac{\partial \vec{F}}{\partial e} = \vec{0}, \quad \frac{\partial \vec{F}}{\partial e} = \begin{pmatrix} 2ade \\ -1 \end{pmatrix}$$

At this point, this becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial c / \partial e \\ \partial d / \partial e \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} \partial c / \partial e \\ \partial d / \partial e \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

6. (pts) In this problem we consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, evaluated by

$$f(x, y) = (e^x \cos y, e^x \sin y)$$

(a) Is this function one-to-one?

No. $f(0, 0) = (1, 0) = f(0, 2\pi)$

(b) Is this function onto?

No. $e^x \neq 0$, so $f(x, y)$ is never $\vec{0}$.

(c) At what points in the domain is this function locally one-to-one?

$$Df = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}, \quad \det(Df) = e^{2x} \neq 0 \text{ for all } (x, y).$$

Df thus is always 1-1, so f is always locally 1-1.

(d) At what points in the domain is this function locally onto?

$\det(Df) \neq 0$ also means Df is always onto.

So f is always locally onto.