EXAM 3

Math 102, 2010-2011 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. __________

2. __________

3. __________

4. __________

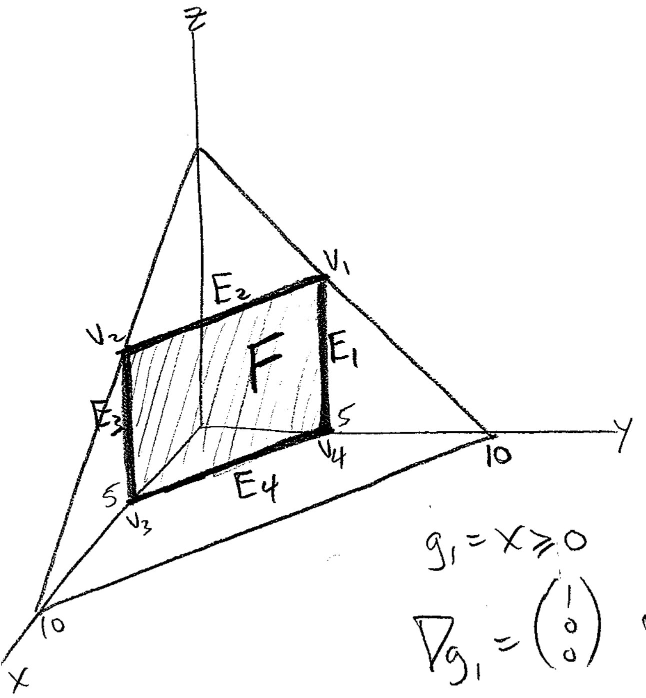
5. __________

6. __________

Total Score __________ (/100 points)

1. *(pts)* The expected annual profit of your business is given by the function $\Pi(x, y, z) = xy + z^2$, where x, y, and z are operational parameters over which you have control, subject to the constraints that $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y + z \leq 10$, and $x + y = 5$.

As CEO of this company, what choices of x, y, and z should you choose?



Domain is shaded region at left, with 1 face (F), 4 edges (E_1, E_2, E_3, E_4), and 4 vertices (V_1, V_2, V_3, V_4)

All 4 vertices are crit pts.

$g_1 = x \geq 0 \quad g_2 = y \geq 0 \quad g_3 = z \geq 0 \quad g_4 = x+y+z \leq 10 \quad h = x+y = 5$

$\nabla g_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \nabla g_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \nabla g_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \nabla g_4 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad Dh = \begin{pmatrix}1\\1\\0\end{pmatrix}$

<u>On F:</u> 1 constraint, $h=5$;

$$\left.\begin{array}{l} \underline{\text{Deg}}: \ Dh = \begin{pmatrix}1\\1\\0\end{pmatrix} \neq 0 \\ \underline{\text{Lagr}}: \ \begin{pmatrix}y\\x\\2z\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow z=0, \text{ not on } F. \end{array}\right\} \text{no c.p.'s}$$

<u>On E_1:</u> 2 constr, $g_1 = 0$, $h=5$

$$\left.\begin{array}{l} \underline{\text{Deg}}: \ Dh = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 0\end{pmatrix} \Leftarrow \text{rank}=2, \text{ no c.p.} \\ \underline{\text{Lagr}}: \ \begin{pmatrix}y\\x\\2z\end{pmatrix} = \mu_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow z=0, \text{ not on } E_1 \end{array}\right\} \text{no c.p.'s}$$

<u>On E_2:</u> 2 constr, $g_4 = 0$, $h=5$

$\underline{\text{Deg}}: \ Dh = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 0\end{pmatrix} \Leftarrow \text{rank}=2, \text{ no c.p.'s}$

$\underline{\text{Lagr}}: \ \begin{pmatrix}y\\x\\2z\end{pmatrix} = \mu_4\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow x=y \Rightarrow x=y=\frac{5}{2},\ z=5$

$\Rightarrow \begin{pmatrix}5/2\\5/2\\5\end{pmatrix}$ is a c.p.

(over →)

(extra space, if needed)

On E_3 : 2 constr., $g_2=0$, $h=5$

Deg: $Dh = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \leftarrow$ rank $=2$, no c.p.

Lagr: $\begin{pmatrix} y \\ x \\ 2z \end{pmatrix} = \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow z=0$, not on E_3

no c.p.'s

On E_4 : 2 constr., $g_3=0$, $h=5$

Deg: $Dh = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \leftarrow$ rank $=2$, no c.p.

Lagr: $\begin{pmatrix} y \\ x \\ 2z \end{pmatrix} = \mu_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow x=y \Rightarrow x=y=\frac{5}{2}$, $z=0$

$\Rightarrow \begin{pmatrix} 5/2 \\ 5/2 \\ 0 \end{pmatrix}$ is a c.p.

5 critical points:

$\Pi \begin{pmatrix} 5/2 \\ 5/2 \\ 5 \end{pmatrix} = \frac{125}{4} \leftarrow$ maximum profit achieved at

$x = 5/2$

$y = 5/2$

$z = 5$

$\Pi \begin{pmatrix} 5/2 \\ 5/2 \\ 0 \end{pmatrix} = \frac{25}{4}$

$\Pi(v_1) = \Pi \begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = 25$

$\Pi(v_2) = \Pi \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} = 25$

$\Pi(v_3) = \Pi \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = 0$

$\Pi(v_4) = \Pi \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = 0$

2. (*pts)* Find the absolute maximizer of the function $f(x, y, z) = 16 - x^2 - y^2 - z^2 - xy - xz - yz$.

$$Df = \begin{pmatrix} -2x & -y & -z \\ -x & -2y & -z \\ -x & -y & -2z \end{pmatrix} = \vec{0}$$

$$\det \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} = -4 \neq 0$$, so $x = y = z = 0$ is only critical point.

$$H = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$ has LPM's: $-2, 3, -4$

So H is negative def and thus also neg. semidef.

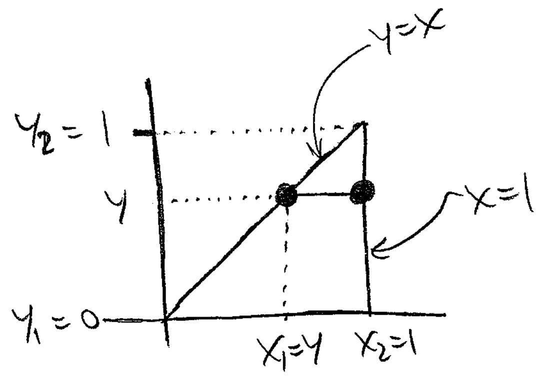
f is defined on $\mathbb{R}^3$ which is open and convex.

So c.p. $(0,0,0)$ is the absolute maximizer.

3. (*pts*)

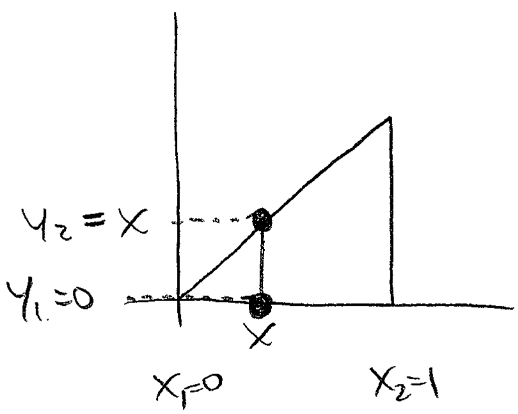
(a) Give a clear description of the domain D for which the double integral $\iint_D e^{-x^2}\, dA$ is calculated by

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy$$



D is the triangle bounded by $y=x$, $x=1$, $y=0$.

(b) Compute the integral above by slicing in the other order.



$$\iint_D e^{-x^2}\, dA = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

$$= \int_0^1 \left(y e^{-x^2} \Big]_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \left(x e^{-x^2} \right) dx$$

Let $u = -x^2$

$du = -2x\, dx$

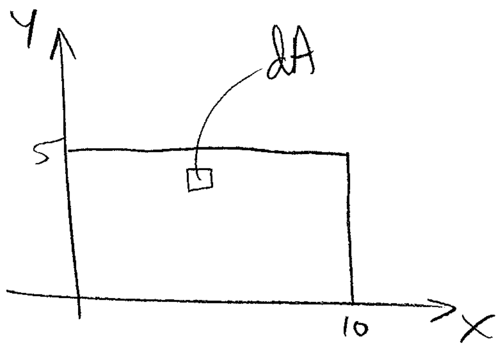
$$= -\frac{1}{2} \int e^u\, du$$

$$= -\frac{1}{2} e^u \Big] = -\frac{1}{2} e^{-x^2} \Big]_0^1 = \boxed{\frac{1}{2}\left(1 - \frac{1}{e}\right)}$$

4. *(pts)* Friendly County is in the shape of a rectangle 10 miles wide (in the east-west direction) and 5 miles tall (in the north-south direction). Locations in Friendly County are given in terms of distances x and y (measured in miles) from the eastern and southern borders (respectively).

The population density (in terms of people per square mile) of Friendly County is given by $\delta(x, y) = 1000(x+y)$. Every person in Friendly County says "hello" a number of times per day that is equal to the square root of the population density at his or her location.

Compute the total number of times per day that the word "hello" is uttered in Friendly County.



$$dP = \#\text{ people in } dA = \delta\, dA$$

$$dH = \#\text{ hello in } dA$$

$$= \left(\frac{\#\text{ hello}}{\text{person}}\right)\left(\#\text{ people in } dA\right)$$

$$= \sqrt{\delta}\; dP = \delta^{3/2}\, dA$$

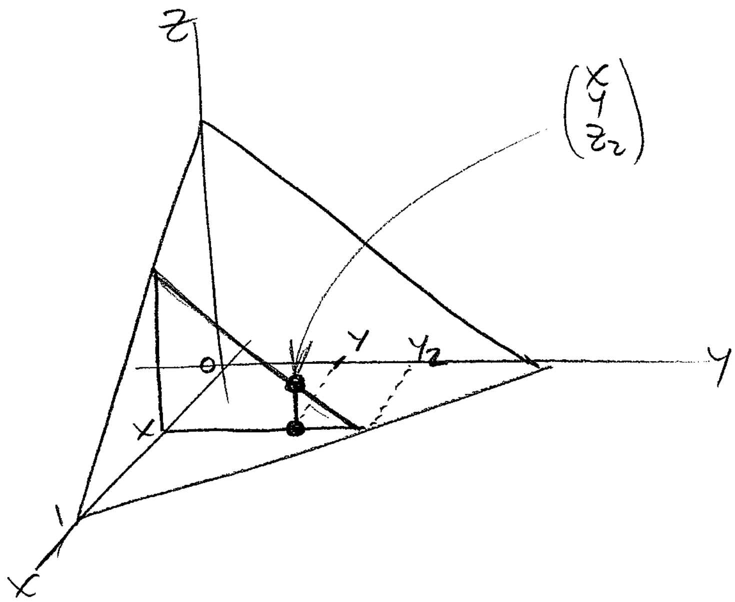
$$H = \iint dH = \iint \delta^{3/2}\, dA = \iint \left(1000\,(x+y)\right)^{3/2} dA = 1000^{3/2} \iint (x+y)^{3/2} dA$$

$$= 1000^{3/2} \int_0^5 \int_0^{10} (x+y)^{3/2}\, dx\, dy = 1000^{3/2} \int_0^5 \left(\frac{2}{5}(x+y)^{5/2} \Big]_{x=0}^{x=10} \right) dy$$

$$= \frac{2 \cdot 1000^{3/2}}{5} \int_0^5 (y+10)^{5/2} - y^{5/2}\, dy = \frac{2}{5} 1000^{3/2} \left(\frac{2}{7}(y+10)^{7/2} - \frac{2}{7} y^{7/2} \right]_0^5$$

$$\boxed{= \frac{4}{35}\, 1000^{3/2} \left(15^{7/2} - 5^{7/2} - 10^{7/2} \right)}$$

5. *(pts)* The probability of three random variables taking values that result in a profit for your company is given by the triple integral of the function e^{-x-y-z} over the domain defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + y + z = 1$. Compute this probability.



Slice 1st $\perp$ x-axis:

x goes from 0 to 1.

Slice next $\perp$ y-axis:

y goes from 0 to y_2

(x, y_2) on $x+y=1$

$\Rightarrow y_2 = 1-x$

Last slice $\perp$ z-axis:

z goes from 0 to z_2; $z_2 = 1-x-y$

$$\int_0^1 \int_0^{1-x} \left(\int_0^{1-x-y} e^{-x-y-z}\, dz \right) dy\, dx = \int_0^1 \int_0^{1-x} \left(-e^{-x-y-z} \Big]_{z=0}^{z=1-x-y} \right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left(-\frac{1}{e} + e^{-x-y} \right) dy\, dx = \int_0^1 \left(-\frac{1}{e} y - e^{-x-y} \Big]_{y=0}^{y=1-x} \right) dx$$

$$= \int_0^1 \left(\left(-\frac{1}{e}\right)(1-x) + \left(e^{-x} - \frac{1}{e}\right) \right) dx = -\frac{1}{e}x + \frac{1}{2}\frac{1}{e}x^2 - e^{-x} - \frac{1}{e}x \Big]_0^1$$

$$= \boxed{1 - \frac{5}{2e}}$$