EXAM 2

Math 103, Summer 2009 Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
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7. __________
8. __________
9. __________
10. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(10 pts)* Compute the acceleration vector for the parametric curve defined by $\vec{x}(t) = (\cos(4t), e^{t^2}, t^2e^t)$.

$$\vec{x} = \begin{pmatrix} \cos 4t \\ e^{t^2} \\ t^2 e^t \end{pmatrix} \qquad \vec{x}' = \begin{pmatrix} -4\sin 4t \\ 2te^{t^2} \\ 2te^t + t^2e^t \end{pmatrix} = \begin{pmatrix} -4\sin 4t \\ 2te^{t^2} \\ (2t+t^2)e^t \end{pmatrix}$$

$$\vec{a} = \vec{x}'' = \begin{pmatrix} -16\cos 4t \\ (2)(e^{t^2}) + (2t)(2te^{t^2}) \\ (2+2t)(e^t) + (2t+t^2)(e^t) \end{pmatrix} = \begin{pmatrix} -16\cos 4t \\ (2+4t^2)e^{t^2} \\ (2+4t+t^2)e^t \end{pmatrix}$$

2. *(10 pts)* Compute $D_{\vec{v}}f(\vec{a})$ directly from the definition, where $f(x,y) = xy^2 + e^x$, $\vec{a} = (3,1)$, and $\vec{v} = (3,4)$.

$$D_{\vec{v}}f(\vec{a}) = \frac{d}{dt}\Big|_{t=0} f(\vec{a} + t\vec{v}) = \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 3+3t \\ 1+4t \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} (3+3t)(1+4t)^2 + e^{3+3t}$$

$$= \frac{d}{dt}\Big|_{t=0} \left(3 + 27t + 72t^2 + 48t^3 + e^{3+3t}\right)$$

$$= \left(27 + 144t + 144t^2 + 3e^{3+3t}\right)\Big|_{t=0}$$

$$= 27 + 3e^3$$

3. *(10 pts)* Use the chain rule to compute the Jacobian matrix of the composition $f \circ g$ in terms of the variables x and y WITHOUT computing the composition function itself. The function f is defined by $f(x, y) = (x^2y - y^2, 3xy^4)$ and the function g is defined by $g(x, y) = (xy, x - y)$.

$$f\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}u^2v - v^2\\ 3uv^4\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} \xrightarrow{g} \begin{pmatrix}u\\v\end{pmatrix} \xrightarrow{f}$$

$$g\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}xy\\ x-y\end{pmatrix} = \begin{pmatrix}u\\v\end{pmatrix}$$

$$J_f = \begin{pmatrix} 2uv & u^2 - 2v \\ 3v^4 & 12uv^3 \end{pmatrix} \qquad J_g \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(xy)(x-y) & (xy)^2 - 2(x-y) \\ 3(x-y)^4 & 12(xy)(x-y)^3 \end{pmatrix}$$

$$J_{f\circ g} = J_f J_g = \begin{pmatrix} 2xy^2(x-y) + x^2y^2 - 2(x-y) & 2x^2y(x-y) - x^2y^2 + 2(x-y) \\ 3y(x-y)^4 + 12xy(x-y)^3 & 3x(x-y)^4 - 12xy(x-y)^3 \end{pmatrix}$$

4. *(10 pts)* Near the point $(1, 2, 1)$ on the surface defined by the equation $3x^2y^3 - xyz^2 = 22$, explain how you know that x can be viewed locally as a function of y and z. Compute $\frac{\partial x}{\partial z}$ at this point.

$$F(x,y,z) = 3x^2y^3 - xyz^2$$

$$\frac{\partial F}{\partial x} = 6xy^3 - yz^2 \qquad \left.\frac{\partial F}{\partial x}\right|_{(1,2,1)} = 48 - 2 = 46 \neq 0$$

Since $\frac{\partial F}{\partial x} \neq 0$, x is locally a fn of y, z.

So $\displaystyle 6xy^3 \frac{\partial x}{\partial z} - \left(\frac{\partial x}{\partial z}\right)(yz^2) - (x)(2yz) = 0$

$$\frac{\partial x}{\partial z} = \frac{2xyz}{6xy^3 - yz^2}$$

$$\left.\frac{\partial x}{\partial z}\right|_{(1,2,1)} = \frac{4}{46} = \boxed{\frac{2}{23}}$$

5. *(10 pts)* Points on Normal Hill have altitude given by the function

$$h(x, y) = 200 - 10(x + y)^2 - 30y^2$$

where x and y represent the horizontal distances east and north (respectively) from a fixed point of reference. Bob is standing on the hill at the point where $x = 1$ and $y = 1$. How steep is it where Bob is standing?

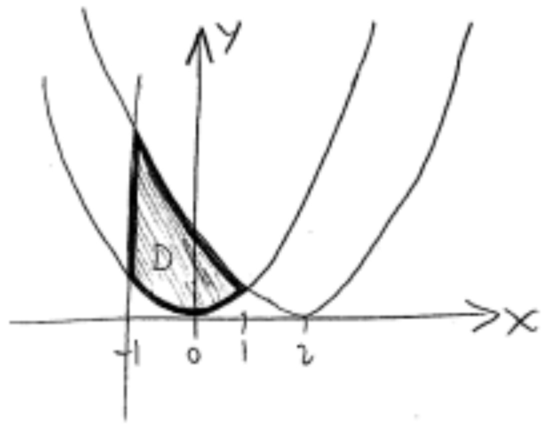
Steepness $= \|\nabla h\|$

$$\nabla h = \begin{pmatrix} -20(x+y) \\ -20(x+y) - 60y \end{pmatrix} \qquad \nabla h\Big|_{(1,1)} = \begin{pmatrix} -40 \\ -100 \end{pmatrix}$$

$$\|\nabla h\| = 10\sqrt{116}$$

6. *(10 pts)* The region D in the xy-plane is bounded by the curves $y = x^2$, $y = (x-2)^2$, and $x + 1 = 0$. The concentration of bacteria at nearby points on the xy-plane is given by $C(x, y) = 1000 - x^2y$. Compute the total number of bacteria in the region D.

$$B = \iint_D C\,dA = \iint_D (1000 - x^2y)\,dA$$



$$= \int_{-1}^{1}\int_{x^2}^{(x-2)^2} (1000 - x^2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left(1000y - \tfrac{1}{2}x^2y^2\right]_{y=x^2}^{y=(x-2)^2} dx = \int_{-1}^{1} (1000)(4-4x) - \left(\tfrac{1}{2}x^2\right)\left((x-2)^4 - x^4\right) dx$$

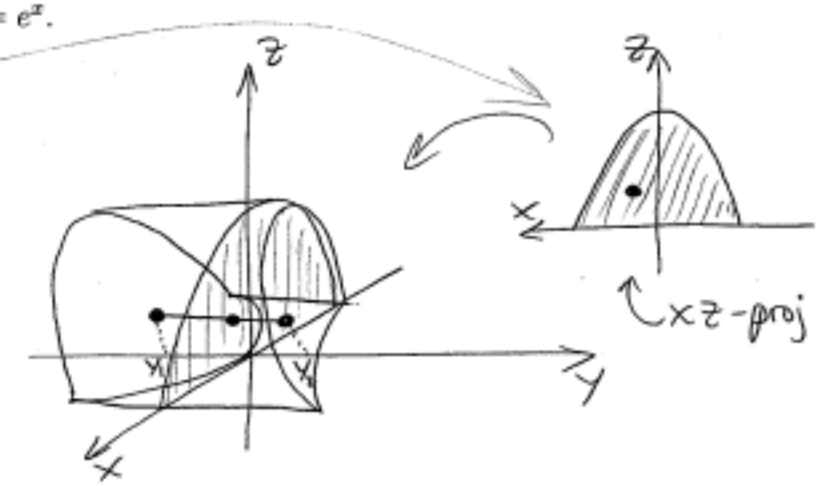
$$= \int_{-1}^{1} 4000 - \underbrace{\cancel{4000x}}_{0 \text{ by symm}} - \left(\tfrac{1}{2}x^2\right)\left(-\underbrace{\cancel{8x^3}}_{0 \text{ by symm}} + 24x^2 - \underbrace{\cancel{32x}}_{0 \text{ by symm}} + 16\right) dx$$

$$= \int_{-1}^{1} 4000 - 12x^4 - 8x^2\,dx$$

$$= 4000x - \frac{12}{5}x^5 - \frac{8}{3}x^3\Big]_{-1}^{1} = 8000 - \frac{24}{5} - \frac{16}{3}$$

7. *(10 pts)* Write down a nested integral (but do not evaluate it) that represents the triple integral of the function $f(x, y, z) = x^2yz$ over the region bounded by the surfaces $z = 0$, $z = 4 - x^2$, $y + x^2 + z^2 = 0$, and $y = e^x$.

Proj. to xz-plane is :



So $x \in [-2, 2]$

$z \in [0, 4 - x^2]$

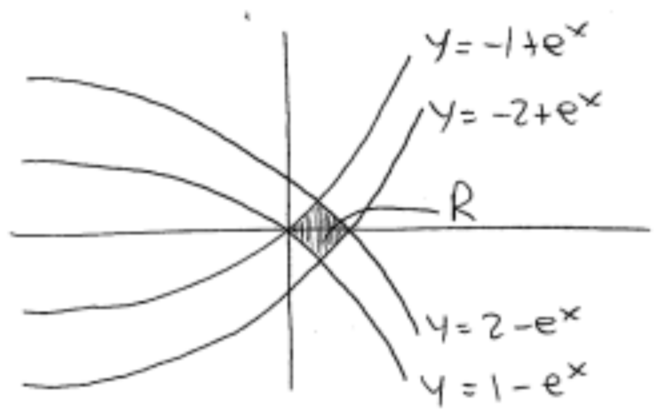
For any x, z,

$(y = -x^2 - z^2 < 0)$ is $<$ $(y = e^x > 0)$

So $$\iiint_R f\,dV = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{-x^2-z^2}^{e^x} x^2yz \, dy\, dz\, dx$$

8. *(10 pts)* Compute the integral $\iint_R y^3 e^x \, dx\, dy$, where R is the region bounded by the four curves $y - e^x = -1$, $y - e^x = -2$, $y + e^x = 1$, and $y + e^x = 2$.

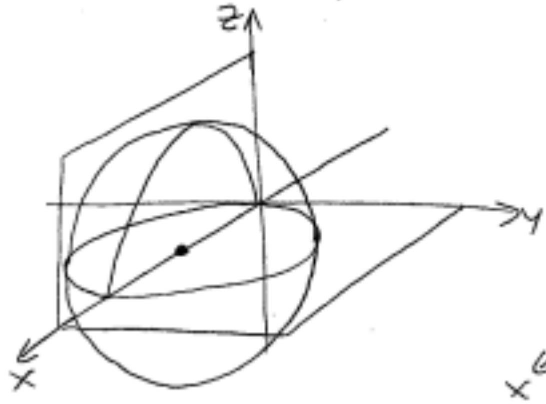
These curves are reflections of each other through the x-axis, so R is symmetric through x-axis.

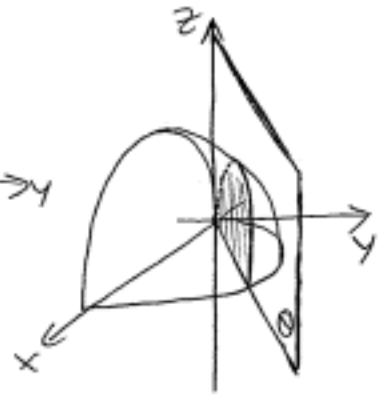


$f(x, -y) = (-y)^3 e^x = -y^3 e^x = -f(x, y)$, so f has odd symm. through x-axis.

$$\Rightarrow \iint = 0 \text{ by symmetry.}$$

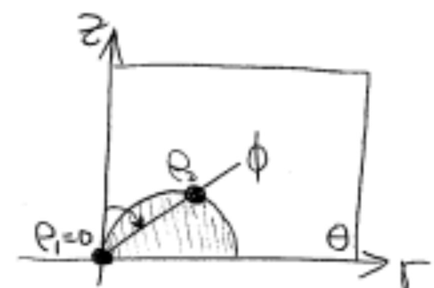
9. *(10 pts)* Write down a nested integral in spherical coordinates (but do not evaluate it) that represents the triple integral of the function $f(x, y, z) = x - y - z$ over the region defined by $(x-5)^2 + y^2 + z^2 \leq 25$, and $x, y, z \geq 0$.





$\Longrightarrow$

$\theta \in [0, \pi/2]$



$\phi \in [0, \pi/2]$

$(x-1)^2 + y^2 + z^2 = 25$

$x^2 - 10x + \cancel{25} + y^2 + z^2 = \cancel{25}$

$\rho^2 - 10\rho\sin\phi\cos\theta = 0$

$\Longrightarrow \rho = 10\sin\phi\cos\theta$

Then

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{10\sin\phi\cos\theta} \big(\rho\sin\phi(\cos\theta - \sin\theta)\big) - \rho\cos\phi\big)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

10. *(10 pts)* Compute the mass of the wire bent into the shape of the curve $y = x^2$ between $x = 0$ and $x = 1$, where the density is given by $\delta(x, y) = 24xy$.

$$m = \int_C dm = \int_C \delta\, ds = \int_a^b \delta \,\|\vec{x}\,'\|\, dt$$

$x = t$ $\quad t \in [0,1]$ $\quad x' = 1$

$y = t^2$ $\quad\quad\quad\quad\quad y' = 2t$

$$m = \int_0^1 24(t)(t^2)\sqrt{1^2 + (2t)^2}\, dt$$

$$= \int_0^1 \underbrace{(t^2)}_{f}\underbrace{\left(24t\sqrt{1+4t^2}\right)}_{g'} dt$$

$f' = 2t \quad g = 2\left(1+4t^2\right)^{3/2}$

$$= 2t^2\left(1+4t^2\right)^{3/2}\Big]_0^1 - \int_0^1 4t\left(1+4t^2\right)^{3/2} dt$$

$u = 1+4t^2$

$du = 8t\,dt$

$$= \left(2t^2\left(1+4t^2\right)^{3/2} - \frac{1}{2}\cdot\frac{2}{5}\left(1+4t^2\right)^{5/2}\right]_0^1$$

$$= \left(2\cdot 5^{3/2} - \frac{1}{5}5^{5/2}\right) - \left(\frac{1}{5}\right)$$

$$= \boxed{5\sqrt{5} - \frac{1}{5}}$$