EXAM 1

Math 103, 2009-2010 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number____________________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

8. __________

9. __________

10. __________

11. __________

12. __________

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(5 pts)* Find the angle between the vectors $(-3, 2, -6)$ and $(3, -12, -4)$.

$$\theta = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right) = \arccos\left(\frac{-9}{7\cdot 13}\right) = \boxed{\arccos\left(\frac{-9}{91}\right)}$$

2. *(10 pts)* Compute the volume of the parallelepiped defined by the vectors $(1, 2, 4)$, $(3, -2, 1)$, $(0, 2, 3)$, and determine if these vectors are listed in right hand order.

$$\det A = \det\begin{pmatrix} 1 & 2 & 4 \\ 3 & -2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = (1)\det\begin{pmatrix} -2 & 1 \\ 2 & 3 \end{pmatrix} - (2)\det\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + (4)\det\begin{pmatrix} 3 & -2 \\ 0 & 2 \end{pmatrix}$$

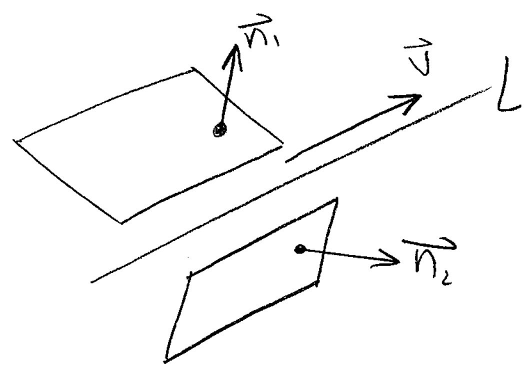
$$= -8 - 18 + 24 = -2$$

$\det < 0 \Rightarrow$ $\boxed{\text{\underline{not} RH order.}}$ $\quad \text{Vol.} = |\det| = \boxed{2}$

3. *(10 pts)* Find the symmetric equations of the line that passes through the point $(2, -6, 3)$ and is parallel to each of the planes $3x - 9y + 4z = 10$ and $2x - y + 4z = 12$.

Choose $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$= \begin{pmatrix} 3 \\ -9 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -32 \\ -4 \\ 15 \end{pmatrix}$$



Line param. by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} + t\begin{pmatrix} -32 \\ -4 \\ 15 \end{pmatrix}$

Symm. eqs are

$$\boxed{\frac{x-2}{-32} = \frac{y+6}{-4} = \frac{z-3}{15}}$$

4. *(10 pts)* Find the spherical equation for the sphere of radius 1 centered at $(0,0,1)_r$.

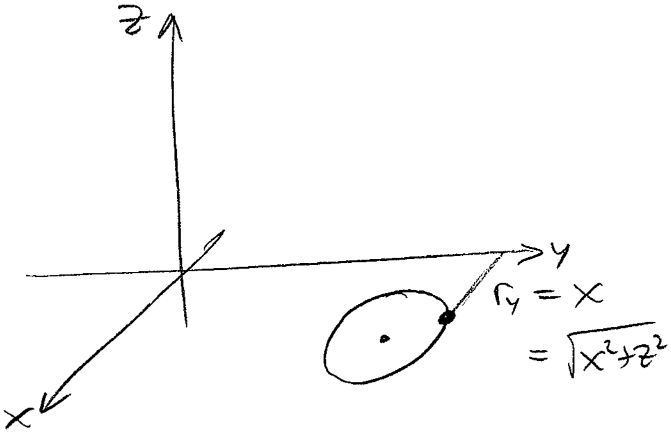
Rect: $x^2+y^2+(z-1)^2=1$

$x^2+y^2+z^2-2z+1=1$

$\rho^2-2\rho\cos\phi=0$

$\boxed{\rho=2\cos\phi}$

5. *(10 pts)* Find the equation for the surface obtained by rotating around the y-axis the circle in the xy-plane with radius 2 centered at $(3,4)$.



Eq. of circle in xy-plane:

$(x-3)^2+(y-4)^2=4$

$x^2-6x+(y-4)^2+5=0$

This is the cross-section of the desired surface by the half plane $\{z=0,\ x\geq 0\}$, on which we have $r_y=\sqrt{x^2+z^2}=x$. Surface eqn then is

$$\boxed{(x^2+z^2)-6\sqrt{x^2+z^2}+(y-4)^2+5=0}$$

For the problems on this page, we consider the plane P with equation $3x - 2y - 4z = 12$.

6. *(5 pts)* The plane P is the graph of a function $f : \mathbb{R}^a \to \mathbb{R}^b$. Find a, b, and an explicit formula for evaluating the function f.

$$z = \frac{3x - 2y - 12}{4}$$ This is the graph $z = f(x,y)$ of $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x,y) = \frac{3x - 2y - 12}{4}$$

7. *(5 pts)* The plane P is a level set of a function $g : \mathbb{R}^c \to \mathbb{R}^d$. Find c, d, and an explicit formula for evaluating such a function g.

$$3x - 2y - 4z - 12 = 0$$ This is the level set $g^{-1}(0)$ of the function $g : \mathbb{R}^3 \to \mathbb{R}^1$ given by

$$g(x,y,z) = 3x - 2y - 4z - 12$$

8. *(5 pts)* The plane P is parametrized by a function $h : \mathbb{R}^p \to \mathbb{R}^q$. Find p, q, and an explicit formula for evaluating such a function h.

The graph parametrization using f above gives us $h : \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$h(u,v) = \left(u, v, \frac{3u - 2v - 12}{4}\right)$$

9. *(10 pts)* The linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ has

$$L\left(\begin{bmatrix}1\\3\\2\end{bmatrix}\right) = \begin{bmatrix}4\\6\\-2\end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix}3\\2\\4\end{bmatrix}\right) = \begin{bmatrix}-3\\1\\3\end{bmatrix}$$

Compute $L\left(\begin{bmatrix}4\\5\\6\end{bmatrix}\right)$. *(Hint: Look for an easy relationship between the input vectors.)*

$$L\begin{pmatrix}4\\5\\6\end{pmatrix} = L\left(\begin{pmatrix}1\\3\\2\end{pmatrix} + \begin{pmatrix}3\\2\\4\end{pmatrix}\right) = L\begin{pmatrix}1\\3\\2\end{pmatrix} + L\begin{pmatrix}3\\2\\4\end{pmatrix}$$

$$= \begin{pmatrix}4\\6\\-2\end{pmatrix} + \begin{pmatrix}-3\\1\\3\end{pmatrix} = \boxed{\begin{pmatrix}1\\7\\1\end{pmatrix}}$$

10. *(10 pts)* Compute the matrix product

$$\begin{pmatrix}4 & 6\\3 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & -3 & 7\\1 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} 4\cdot 2 + 6\cdot 1 & 4\cdot 3 + 6\cdot 0 & 4\cdot 7 + 6\cdot 1 \\ 3\cdot 2 + 2\cdot 1 & 3\cdot 3 + 2\cdot 0 & 3\cdot 7 + 2\cdot 1 \\ 2\cdot 2 + -1\cdot 1 & 2\cdot 3 + -1\cdot 0 & 2\cdot 7 + -1\cdot 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix}14 & -12 & 34\\8 & -9 & 23\\3 & -6 & 13\end{pmatrix}}$$

11. *(10 pts)* Consider the parametric curve $\vec{x}(t) = (\cos t, \sin t, t)$, which is a helix around the z-axis. The closest point on the z-axis to $\vec{x}(t)$ is $\vec{c}(t) = (0, 0, t)$. Show that $\vec{x}'$ is always orthogonal to $\vec{x} - \vec{c}$.

$$\vec{x}' = (-\sin t, \cos t, 1)$$

$$\vec{x} - \vec{c} = (\cos t, \sin t, 0)$$

$$\vec{x}' \cdot (\vec{x} - \vec{c}) = -\sin t \cos t + \sin t \cos t + 0 = 0$$

So $\vec{x}'$ and $\vec{x} - \vec{c}$ are orthogonal.

12. *(10 pts)* The air pollution (measured in ppm) near a factory is given by the function $p(x, y) = 600e^{-(x^2+y^2)/16}$, where x and y are measured in miles east and north from the plant (respectively). Bob is at the point $(2, 1)$ and is moving north at 2 miles per hour. What is the rate of change of the air pollution that Bob is breathing?

$$\vec{a} = (2,1), \quad \vec{v} = (0,2), \quad \frac{dp}{dt} = D_{\vec{v}}\, p(\vec{a})$$

$$D_{\vec{v}}\, p(\vec{a}) = \frac{d}{dt}\Big|_{t=0} p(\vec{a} + t\vec{v})$$

$$= \frac{d}{dt}\Big|_{t=0} p\begin{pmatrix} 2 \\ 1+2t \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} 600\, e^{-(2^2 + (1+2t)^2)/16}$$

$$= \left(600 \left(e^{-(4t^2+4t+5)/16}\right)\left(-\frac{8t+4}{16}\right)\right)\Big|_{t=0}$$

$$= 600\left(e^{-5/16}\right)\left(-\frac{1}{4}\right)$$

$$= \boxed{-150e^{-5/16}}$$