EXAM 1

Math 103, 2010 Summer Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

ID number ________________

1. ________
2. ________
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8. ________
9. ________
10. ________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

Total Score ________ (/100 points)

1. *(10 pts)* Given the vectors $\vec{v} = (3, 2, 6)$ and $\vec{w} = (-12, 3, 4)$, find the value of $\text{comp}_{\vec{v}}(\vec{w})$.

$$\text{comp}_{\vec{v}}(\vec{w}) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|} = \frac{-36+6+24}{\sqrt{3^2+2^2+6^2}} = \boxed{-\frac{6}{7}}$$

2. *(10 pts)* Compute the area of the parallelogram defined by the vectors $\vec{a} = (5, 3, -4)$ and $\vec{b} = (2, 1, 4)$, and determine if $\vec{a}$, $\vec{b}$, $\vec{e}_3$ are listed in right hand order.

$$\text{area} = \|\vec{a}\times\vec{b}\| = \|(16, -28, -1)\| = \sqrt{16^2+28^2+1^2} = \sqrt{1041}$$

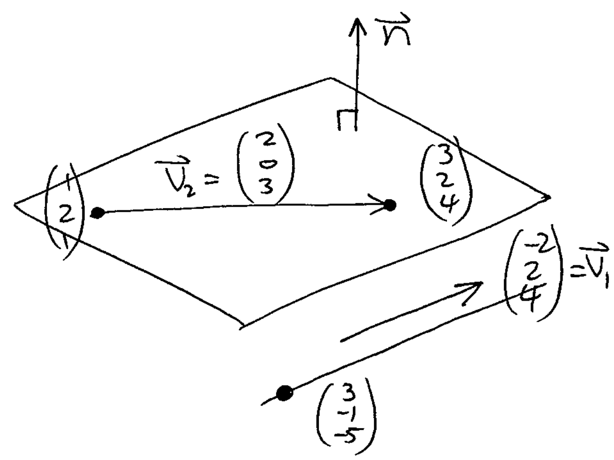
$$\det\begin{pmatrix} \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{b}\ \text{---} \\ \text{---}\ \vec{e}_3\ \text{---} \end{pmatrix} = \det\begin{pmatrix} \text{---}\ \vec{e}_3\ \text{---} \\ \text{---}\ \vec{a}\ \text{---} \\ \text{---}\ \vec{b}\ \text{---} \end{pmatrix} = \vec{e}_3\cdot(\vec{a}\times\vec{b}) = -1 < 0$$

So $\vec{a}, \vec{b}, \vec{e}_3$ are not in R.H.O.

3. *(10 pts)* Find the equation of the plane that passes through the points $\vec{p} = (1, 2, 1)$ and $\vec{q} = (3, 2, 4)$, and is parallel to the line parametrized by $\vec{x}(t) = (3 - 2t, 2t - 1, 4t - 5)$.

$\vec{n} \perp \vec{v}_1, \vec{v}_2$, so choose

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ -4 \end{pmatrix}$$

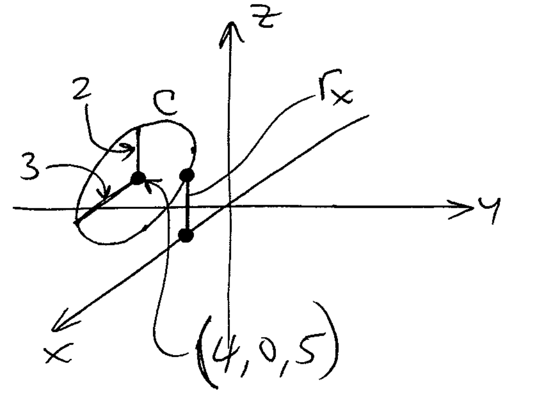


Choose $\vec{x}_0 = \vec{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in plane.

Eq. is $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0$

$$\boxed{6x + 14y - 4z = 30}$$

4. *(10 pts)* Let C be the ellipse in the xz-plane in $\mathbb{R}^3$ with vertices at the points $(1,0,5)$, $(7,0,5)$, $(4,0,3)$, $(4,0,7)$. S is the surface obtained by rotating C around the x-axis. Find the equation for S.



Eq. of C in xz-plane is

$$\left(\frac{x-4}{3}\right)^2 + \left(\frac{z-5}{2}\right)^2 = 1$$

This in upper half of xz-plane $(z>0)$, so $r_x = z$. Rotating, we have $r_x = \sqrt{y^2+z^2}$, so eq. becomes

$$\boxed{\left(\frac{x-4}{3}\right)^2 + \left(\frac{\sqrt{y^2+z^2}-5}{2}\right)^2 = 1}$$

5. *(10 pts)* Write down an explicit formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ for which $T(1,1) = (3,2,5)$ and $T(1,2) = (4,3,1)$.

$$T(0,1) = T\big((1,2)-(1,1)\big) = T(1,2) - T(1,1) = \begin{pmatrix}4\\3\\1\end{pmatrix} - \begin{pmatrix}3\\2\\5\end{pmatrix} = \begin{pmatrix}1\\1\\-4\end{pmatrix}$$

$$T(1,0) = T\big((1,1)-(0,1)\big) = T(1,1) - T(0,1) = \begin{pmatrix}3\\2\\5\end{pmatrix} - \begin{pmatrix}1\\1\\-4\end{pmatrix} = \begin{pmatrix}2\\1\\9\end{pmatrix}$$

$$T(x,y) = xT(1,0) + yT(0,1) = x\begin{pmatrix}2\\1\\9\end{pmatrix} + y\begin{pmatrix}1\\1\\-4\end{pmatrix}$$

$$= \begin{pmatrix}2x+y\\x+y\\9x-4y\end{pmatrix}$$

For the problems on this page, we consider the surface S with equation $xy^2 = x^2 + 2z$.

6. *(10 pts)* The surface S is the graph of a function $f : \mathbb{R}^a \to \mathbb{R}^b$. Find a, b, and an explicit formula for evaluating the function f.

$$xy^2 = x^2 + 2z$$

$$z = \frac{1}{2}\left(xy^2 - x^2\right)$$

This is the graph $z = f(x,y)$ of $\boxed{f : \mathbb{R}^2 \to \mathbb{R}^1}$ given by

$$\boxed{f(x,y) = \frac{1}{2}\left(xy^2 - x^2\right)}$$

7. *(10 pts)* The surface S is a level set of a function $g : \mathbb{R}^c \to \mathbb{R}^d$. Find c, d, and an explicit formula for evaluating such a function g.

$$xy^2 = x^2 + 2z$$

$$xy^2 - x^2 - 2z = 0$$

This is the levelset $g = 0$ of $\boxed{g : \mathbb{R}^3 \to \mathbb{R}^1}$ given by

$$\boxed{g(x,y,z) = xy^2 - x^2 - 2z}$$

8. *(10 pts)* The surface S is parametrized by a function $h : \mathbb{R}^p \to \mathbb{R}^q$. Find p, q, and an explicit formula for evaluating such a function h.

Using the graph parametrization from #6, we get

$$\boxed{h(u,v) = \left(u, v, \frac{1}{2}\left(uv^2 - u^2\right)\right)} = (x,y,z)$$

$$\boxed{h : \mathbb{R}^2 \to \mathbb{R}^3}$$

9. *(10 pts)* A particle has position given by $\vec{x}(t)$. It is known that it starts at rest at the origin, and that its acceleration is given by $\vec{a}(t) = (\cos t, \sin 2t, e^t)$ while $t \in [0, \pi]$. Compute $\vec{x}(1)$.

$$\vec{v} = \int \vec{a}\, dt = \int \begin{pmatrix} \cos t \\ \sin 2t \\ e^t \end{pmatrix} dt = \begin{pmatrix} \sin t + c_1 \\ -\frac{1}{2}\cos 2t + c_2 \\ e^t + c_3 \end{pmatrix}$$

$$t=0 \Rightarrow \vec{v} = \vec{0}, \quad \text{so} \quad c_1 = 0,\ c_2 = \tfrac{1}{2},\ c_3 = -1$$

$$\vec{x} = \int \vec{v}\, dt = \int \begin{pmatrix} \sin t \\ -\frac{1}{2}\cos 2t + \frac{1}{2} \\ e^t - 1 \end{pmatrix} dt = \begin{pmatrix} -\cos t + c_4 \\ -\frac{1}{4}\sin 2t + \frac{1}{2}t + c_5 \\ e^t - t + c_6 \end{pmatrix}$$

$$t=0 \Rightarrow \vec{x} = \vec{0}, \quad \text{so} \quad c_4 = 1,\ c_5 = 0,\ c_6 = -1$$

$$\vec{x}(t) = \begin{pmatrix} -\cos t + 1 \\ -\frac{1}{4}\sin 2t + \frac{1}{2}t \\ e^t - t - 1 \end{pmatrix} \qquad \boxed{\vec{x}(1) = \begin{pmatrix} 1 - \cos(1) \\ \frac{1}{2} - \frac{1}{4}\sin(2) \\ e - 2 \end{pmatrix}}$$

10. *(10 pts)* The moving point $\vec{x} = (x, y)$ has velocity $(3, 2)$ at the moment that it passes through the point $(1, 4)$. At that moment, what is the rate of change of the function given by $f(x, y) = (2x^3y, e^{xy})$?

$$\vec{v} = (3,2) \qquad \vec{a} = (1,4)$$

$$\frac{df}{dt} = D_{\vec{v}} f(\vec{a})$$

$$= \frac{d}{dt}\Big|_{t=0} f\big((1,4) + t(3,2)\big)$$

$$= \frac{d}{dt}\Big|_{t=0} f(1+3t,\ 4+2t)$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 2(1+3t)^3(4+2t) \\ e^{(1+3t)(4+2t)} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} (1+9t+27t^2+27t^3)(8+4t) \\ e^{4+14t+6t^2} \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 8+76t+252t^2+324t^3+108t^4 \\ e^{4+14t+6t^2} \end{pmatrix} = \begin{pmatrix} 76+504t+972t^2+432t^3 \\ (14+12t)e^{4+14t+6t^2} \end{pmatrix}\Bigg|_{t=0}$$

$$= \boxed{(76, 14e^4)}$$