EXAM 2

Math 103, 2012 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

9. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score ____________ (/100 points)

1. *(5 pts)* The regularity condition P is said to be "stronger" than the condition Q if every function satisfying P also satisfies Q. For each of the following pairs of regularity conditions, circle the one which is *stronger*.

(a)	differentiability	directional differentiability
(b)	partial differentiability	directional linearity
(c)	continuous differentiability	directional linearity
(d)	continuous differentiability	differentiability
(e)	partial differentiability	directional differentiability

2. *(12 pts)* Compute the directional derivative of the function $f(x,y,z) = (e^{x^2y^2z^3}, \sin(\pi x y^3))$ at the point $\underbrace{(1,1,1)}_{\vec{a}}$ with velocity $\underbrace{(2,3,4)}_{\vec{v}}$, by any method from this course.

$$D_{\vec{v}} f(\vec{a}) = D_{f,\vec{a}}(\vec{v}) = J_{f,\vec{a}}\vec{v}$$

$$J_f = \begin{pmatrix} 2xy^2z^3 e^{x^2y^2z^3} & 2yx^2z^3 e^{x^2y^2z^3} & 3z^2x^2y^2 e^{x^2y^2z^3} \\ \pi y^3 \cos(\pi x y^3) & 3\pi x y^2 \cos(\pi x y^3) & 0 \end{pmatrix}$$

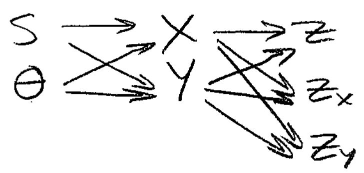
$$J_{f,\vec{a}} = \begin{pmatrix} 2e & 2e & 3e \\ -\pi & -3\pi & 0 \end{pmatrix}$$

$$D_{\vec{v}} f(\vec{a}) = \begin{pmatrix} 2e & 2e & 3e \\ -\pi & -3\pi & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} 22e \\ -11\pi \end{pmatrix}}$$

3. *(12 pts)* In this problem we consider a system of coordinates for $\mathbb{R}^2\backslash\{\vec{0}\}$, using coordinates s and θ, defined by the equations

$$\begin{aligned} x &= e^s \cos\theta \\ y &= e^s \sin\theta \end{aligned}$$

The variable z is a known function of x and y. Derive a formula for $\dfrac{\partial^2 z}{\partial s \partial \theta}$ in terms of x, y, and the various partials of z with respect to x and y.



$$\frac{\partial x}{\partial s} = e^s\cos\theta = x \qquad \frac{\partial x}{\partial \theta} = -e^s\sin\theta = -y$$

$$\frac{\partial y}{\partial s} = e^s\sin\theta = y \qquad \frac{\partial y}{\partial \theta} = e^s\cos\theta = x$$

$$\frac{\partial z}{\partial s} = z_x \frac{\partial x}{\partial s} + z_y \frac{\partial y}{\partial s} = z_x x + z_y y$$

$$\begin{aligned} \frac{\partial^2 z}{\partial s\,\partial\theta} &= \left(\frac{\partial z_x}{\partial\theta} x + z_x \frac{\partial x}{\partial\theta}\right) + \left(\frac{\partial z_y}{\partial\theta} y + z_y \frac{\partial y}{\partial\theta}\right) \\ &= \left(\frac{\partial z_x}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z_x}{\partial y}\frac{\partial y}{\partial\theta}\right)x + z_x(-y) + \left(\frac{\partial z_y}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z_y}{\partial y}\frac{\partial y}{\partial\theta}\right)y + z_y(x) \\ &= \left(z_{xx}(-y) + z_{xy}(x)\right)x + z_x(-y) + \left(z_{xy}(-y) + z_{yy}(x)\right)y + z_y(x) \\ &= \boxed{-xy\, z_{xx} + (x^2-y^2) z_{xy} + xy\, z_{yy} - y z_x + x z_y} \end{aligned}$$

4. *(12 pts)* On the surface $\underbrace{e^{x+y} + e^{y+z}}_{F(x,y,z)} = 2e^3$, at the point $(1,2,1)$, what is the value of $\dfrac{\partial x}{\partial y}$?

$$\frac{\partial F}{\partial x} = e^{x+y} + 0 = e^{x+y} \qquad \frac{\partial F}{\partial x}(1,2,1) = e^3 \neq 0$$

So we can view x locally as a function of y and z.
Taking a partial w.r.t. y in the original equation, we get

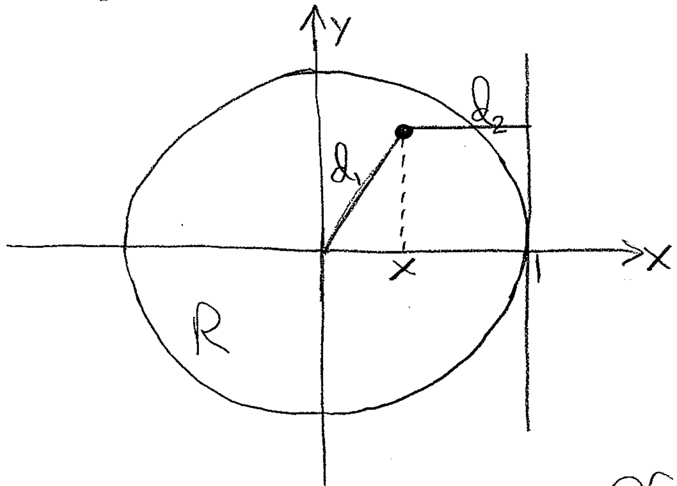
$$e^{x+y}\frac{\partial}{\partial y}(x+y) + e^{y+z}\frac{\partial}{\partial y}(y+z) = 0$$

$$e^{x+y}\left(\frac{\partial x}{\partial y} + 1\right) + e^{y+z}(1) = 0$$

$$\frac{\partial x}{\partial y} = -\frac{e^{y+z}}{e^{x+y}} - 1 = \boxed{-e^{z-x} - 1}$$

5. *(15 pts)* Locations in a particular city are referred to by x and y coordinates, measured in miles. The city is bounded by the unit circle in the xy-plane, with downtown at the origin. There is a river running along the line $x = 1$. Average property values at the location (x, y) is given by the function $p(x, y) = 3ke^{-d_1} + k(2 - d_2)$, where d_1 is the distance to downtown, d_2 is the distance to the river, and $k = \$100,000/\text{acre} = \$6.4M/\text{square mile}$.

Compute the total value of all of the land in this city.



$$d_1 = \sqrt{x^2+y^2}$$

$$d_2 = 1-x$$

$$p(x,y) = 3ke^{-\sqrt{x^2+y^2}} + k(1+x)$$

$$\text{value} = \iint_R p(x,y)\,dA = \iint_R 3ke^{-\sqrt{x^2+y^2}} + kx + k\,dA$$

$$= \iint_R 3ke^{-r}\,r\,dr\,d\theta + \iint_R kx\,dx\,dy + \iint_R k\,dA$$

$$3k\int_0^{2\pi}\int_0^1 re^{-r}\,dr\,d\theta$$

$$= 6\pi k\int_0^1 re^{-r}\,dr \quad \leftarrow \quad f=r \quad g'=e^{-r} \quad f'=1 \quad g=-e^{-r}$$

$$= 6\pi k\left(-re^{-r} - \int (1)(-e^{-r})\,dr\right]_0^1$$

$$= 6\pi k\left(-re^{-r} - e^{-r}\right]_0^1$$

$$= 6\pi k\left((-2e^{-1}) - (-1)\right)$$

$$= 6\pi k\left(1 - \frac{2}{e}\right)$$

$f(x,y) = kx$ has odd symmetry over y-axis.

R is symmetric over y-axis.

$\int = 0$ by symmetry

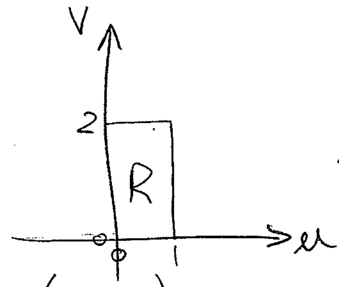
$$= k\iint_R 1\,dA$$

$$= (k)(\text{area of } R)$$

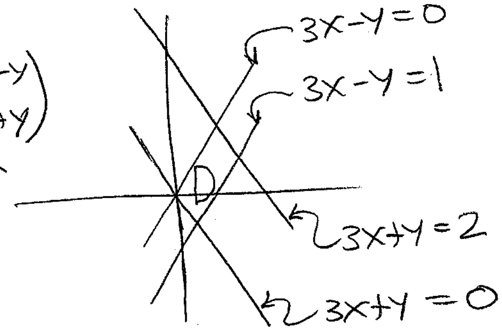
$$= \pi k$$

$$\text{value} = \boxed{6\pi k\left(1-\frac{2}{e}\right) + \pi k}$$

6. *(12 pts)* Use the method of change of variables to compute the value of the integral $\iint_D 6x\,dx\,dy$, where D is the region of the xy-plane bounded by the lines $3x - y = 0$, $3x - y = 1$, $3x + y = 0$, $3x + y = 2$.



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3x-y \\ 3x+y \end{pmatrix}$$



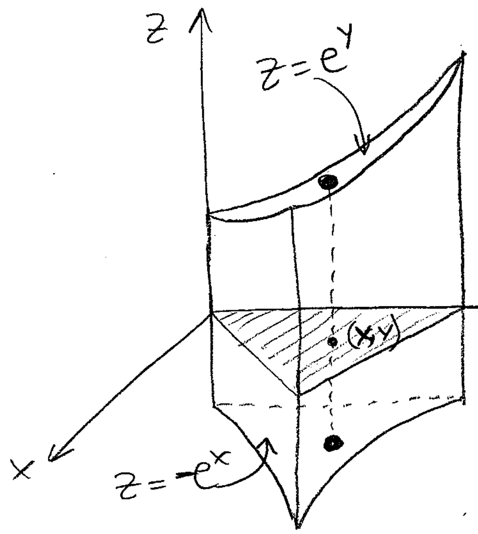
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} = 6$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 1/6$$

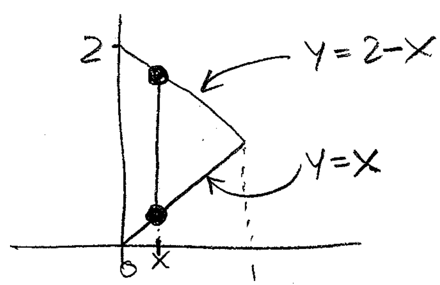
$$\iint_D 6x\,dx\,dy = \iint_R (6x)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^1\int_0^2 (u+v)\left(\tfrac{1}{6}\right) du\,dv$$

$$= \frac{1}{6}\int_0^1 \left(\tfrac{1}{2}u^2 + uv\right]_{u=0}^{u=2} dv = \frac{1}{6}\int_0^1 2+2v\,dv = \frac{1}{6}\left(2v+v^2\right]_0^1 = \boxed{\frac{1}{2}}$$

7. *(12 pts)* Set up as a single triple nested integral (but do not evaluate!) the integral of the function f on the domain bounded by the surfaces $x = 0$, $y = x$, $x + y = 2$, $z = -e^x$, and $z = e^y$.



Projection to xy-plane is



x ranges from 0 to 1

w/fixed x, y ranges from x to 2−x

Then for fixed x and y, z ranges from $-e^x$ to e^y

So
$$\boxed{\iiint_D f\,dV = \int_0^1\int_x^{2-x}\int_{-e^x}^{e^y} f\,dz\,dy\,dx}$$

8. *(10 pts)* Compute the line integral of the function $f(x,y) = xe^{-y}$ along the part of the curve $e^y = \cos(x)$ between $x = 0$ and $x = \pi/4$.

$y = \ln(\cos x)$, graph param: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ \ln(\cos t) \end{pmatrix}$, $t \in [0, \pi/4]$

$\vec{v} = \begin{pmatrix} 1 \\ -\tan t \end{pmatrix} \Rightarrow \|\vec{v}\| = \sqrt{1+\tan^2 t}$
$= \sec t$

$$\int_C f\,ds = \int_0^{\pi/4} x e^{-y} \|\vec{v}\|\,dt = \int_0^{\pi/4} \left(t\, e^{-(\ln\cos t)}\right)(\sec t)\,dt$$

$$= \int_0^{\pi/4} t \sec^2 t\,dt$$

$f = t \quad g' = \sec^2 t$
$f' = 1 \quad g = \tan t$

$$= \left(t \tan t - \int (1) \tan t\,dt\right]_0^{\pi/4}$$

$$= \left(t \tan t + \ln(\cos t)\right)\Big]_0^{\pi/4} = \left(\frac{\pi}{4}\cdot 1 + \ln(\sqrt{1/2})\right) - (0\cdot 0 + 0)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$$

9. *(10 pts)* Compute the surface integral of the function $g(x,y,z) = x^2$ on the unit sphere.

spherical graph param. of $\rho = 1$: $\vec{X} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$ $\quad \phi \in [0,\pi]$, $\theta \in [0, 2\pi]$

$$\vec{X}_\phi = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix} \quad \vec{X}_\theta = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix} \quad \vec{N} = \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \sin\phi\cos\phi \end{pmatrix}$$

$$\|\vec{N}\| = \sin\phi \left\| \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \right\| = \sin\phi\sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi} = \sin\phi$$

$$\iint_S x^2\,dS = \int_0^{2\pi}\int_0^{\pi} (\sin^2\phi\cos^2\theta)(\sin\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \int_0^{\pi} (1-\cos^2\phi)(\sin\phi\,d\phi)\,d\theta \quad \text{let } u = \cos\phi,\ du = -\sin\phi\,d\phi$$

$$= \int_0^{2\pi} \cos^2\theta \left(\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} d\theta = \frac{4}{3}\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta = \boxed{\frac{4\pi}{3}}$$