

EXAM 2

Math 212, 2013 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

9. _____

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: _____

Total Score _____ (/100 points)

1. (8 pts) For the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined by $f(x, y) = \sin(x - y - xy)$ and the point $\vec{a} = (0, 0)$, compute directly from the definition of directional derivative to find a formula for

$$D_{\vec{v}}f(\vec{a})$$

in terms of the coordinates of the vector $\vec{v} = (v_1, v_2)$.

$$\begin{aligned} D_{\vec{v}}f(\vec{a}) &= \left. \frac{d}{dt} f(\vec{a} + t\vec{v}) \right|_{t=0} = \left. \frac{d}{dt} f \begin{pmatrix} 0 + tv_1 \\ 0 + tv_2 \end{pmatrix} \right|_{t=0} \\ &= \left. \frac{d}{dt} \sin(tv_1 - tv_2 - t^2v_1v_2) \right|_{t=0} \\ &= \left. \left[\cos(tv_1 - tv_2 - t^2v_1v_2) (v_1 - v_2 - 2tv_1v_2) \right] \right|_{t=0} \\ &= \boxed{v_1 - v_2} \end{aligned}$$

2. (8 pts) Make an argument using differentiability to compute the same directional derivatives as in the previous question.

$$\text{Partials of } f : \frac{\partial f}{\partial x} = \cos(x - y - xy)(1 - y) \quad \frac{\partial f}{\partial y} = \cos(x - y - xy)(-1 - x)$$

These are continuous, so f is cont. diff and thus diff'bl.

$$\text{Then } D_{\vec{v}}f(\vec{a}) = D_{f, \vec{a}}(\vec{v}) = J_{f, \vec{a}} \vec{v}$$

$$= \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) \Big|_{(0,0)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= (1 \quad -1) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boxed{v_1 - v_2}$$

3. (10 pts) Find a vector that is orthogonal to the graph of the function defined by $f(x, y) = x^2 - y^3$ at the point on this graph with $x = 2$ and $y = 1$.

Graph of f is $z = x^2 - y^3$, which is a level set of $g(x, y, z) = x^2 - y^3 - z$.

So $\nabla g = \begin{pmatrix} 2x \\ -3y^2 \\ -1 \end{pmatrix}$ is orthogonal to the surface.

$$x=2, y=1 \Rightarrow z=3$$

At this point, ∇g is $\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}$

4. (10 pts) Suppose we know that $xy - x^2z + y^3z^2 = 1$. At the point $(1, 1, 1)$ on this surface, can you show that $\frac{\partial x}{\partial y}$ exists, and if so, what value is it?

$$F(x, y, z) = xy - x^2z + y^3z^2$$

$$\frac{\partial F}{\partial x} = y - 2xz \quad \text{so} \quad \frac{\partial F}{\partial x} \Big|_{(1,1,1)} = -1 \neq 0$$

So x can be viewed locally as a fn of y, z .

Viewing x so in the given equation and taking partial w.r.t. y :

$$\frac{\partial x}{\partial y} y + x \left(\frac{\partial y}{\partial y} \right) - 2xz \frac{\partial x}{\partial y} + 3y^2z^2 = 0$$

$$\frac{\partial x}{\partial y} = \frac{-x - 3y^2z^2}{y - 2xz}$$

$$\frac{\partial x}{\partial y} \Big|_{(1,1,1)} = 4$$

5. (12 pts) Let z be a smooth function of x and y , which are the usual functions of r and θ . Compute the following in terms of r , θ , and the various partial derivatives of z with respect to x and y .

$$\frac{\partial}{\partial r} [xz_x \cos(\theta)]$$

$$\begin{array}{ccc} r & \rightarrow & x \\ \theta & \rightarrow & y \end{array} \quad \rightarrow \quad \begin{array}{c} z \\ z_x \end{array}$$

$$\frac{\partial}{\partial r} [xz_x \cos \theta] = \frac{\partial x}{\partial r} z_x \cos \theta + x \frac{\partial z_x}{\partial r} \cos \theta + x z_x \frac{\partial \cos \theta}{\partial r}$$

$$= z_x \cos^2 \theta + x \cos \theta \left[\frac{\partial z_x}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z_x}{\partial y} \frac{\partial y}{\partial r} \right]$$

$$= z_x \cos^2 \theta + x \cos \theta [z_{xx} \cos \theta + z_{xy} \sin \theta]$$

$$= z_x \cos^2 \theta + z_{xx} r \cos^3 \theta + z_{xy} r \cos^2 \theta \sin \theta$$

6. (13 pts) Write down, but do not evaluate, a triple nested integral that represents the integral of the function $f(x, y, z) = xz^3$ over the domain that is inside the sphere $x^2 + y^2 + z^2 = 25$ and above the plane $z = 3$.

Proj. to xy -plane defined
by intersection:

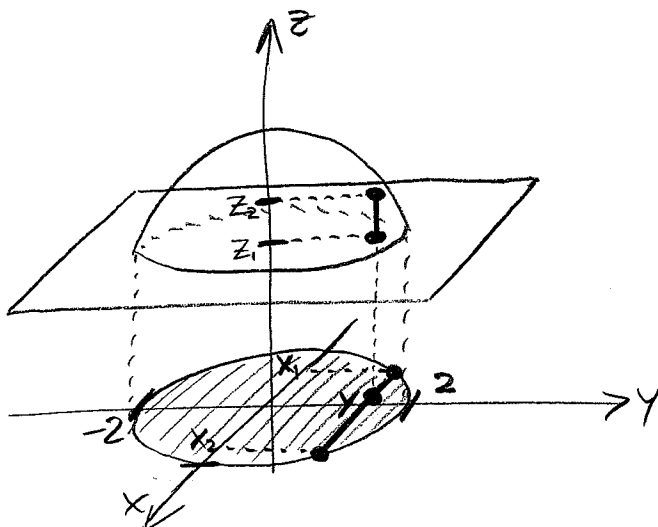
$$x^2 + y^2 + (3)^2 = 25$$

$$x^2 + y^2 = 16$$

on which $y \in [-2, 2]$

and for a given y , $x \in [-\sqrt{4-y^2}, \sqrt{4-y^2}]$

For given x, y , z bounds are determined by the two surfaces.



So

$$\iiint_D xz^3 \, dV = \int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_3^{\sqrt{25-x^2-y^2}} (xz^3) \, dz \, dx \, dy$$

Bonus: (3pts) What is the value of the above integral? Explain in detail.

Reflection through yz -plane: $R(x, y, z) = (-x, y, z)$

And $f(R(\vec{x})) = (-x)(z)^3 = -xz^3 = -f(\vec{x})$.

So f has odd symmetry through this plane; and D is symmetric through same plane. So integral is zero by symmetry.

7. (13 pts) Compute integral below on the domain D bounded by $y = x$, $y = -x$, $x + y = 1$, and $x - y = 1$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2}$$

$\iint_D x \, dx \, dy$

$u = x+y$
 $v = x-y$

Then:

$$\iint_D x \, dx \, dy = \int_0^1 \int_0^1 \left(\frac{u+v}{2} \right) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv = \int_0^1 \int_0^1 \frac{u+v}{4} \, du \, dv$$

$$= \int_0^1 \left(\frac{u^2}{8} + \frac{uv}{4} \right) \Big|_{u=0}^{u=1} \, dv = \int_0^1 \frac{1}{8} + \frac{v}{4} \, dv$$

$$= \left(\frac{v}{8} + \frac{v^2}{8} \right) \Big|_{v=0}^{v=1} = \boxed{\frac{1}{4}}$$

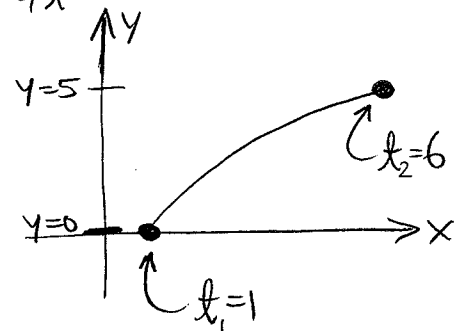
8. (13 pts) Write down, but do not evaluate, a single variable integral that represents the line integral of the function $f(x, y, z) = e^{xy}$ on the curve parametrized by $\vec{x}(t) = (t^2, t-1)$ between $y = 0$ and $y = 5$.

$$\vec{v} = \vec{x}' = \begin{pmatrix} 2t \\ 1 \end{pmatrix} \quad \text{so} \quad \|\vec{v}\| = \sqrt{1+4t^2}$$

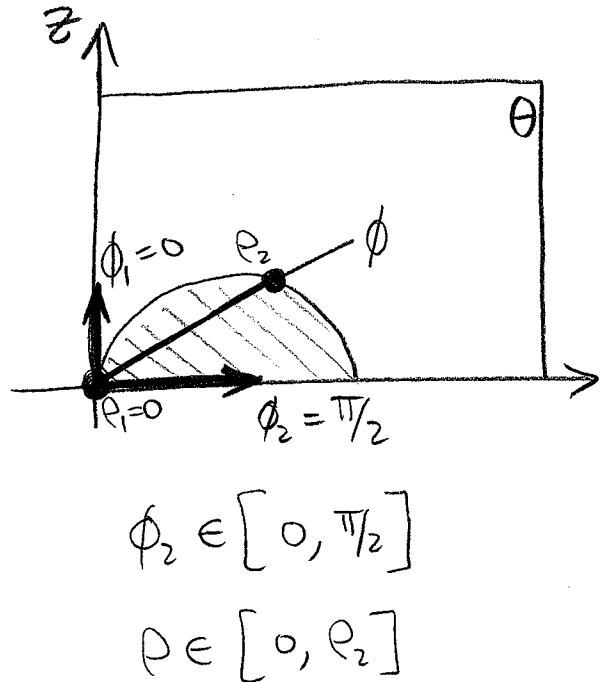
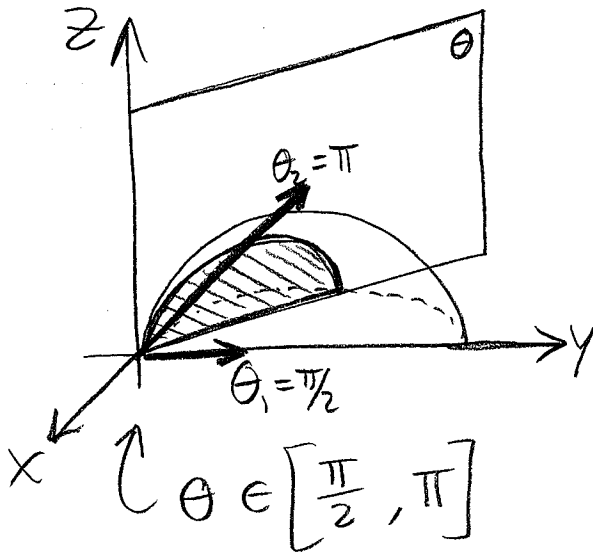
$$\int_C f \, ds = \int_{t_1}^{t_2} e^{xy} \|\vec{v}\| \, dt$$

$$= \int_1^6 e^{(t^2)(t-1)} (\sqrt{1+4t^2}) \, dt$$

$$= \boxed{\int_1^6 e^{t^3-t^2} \sqrt{1+4t^2} \, dt}$$



9. (13 pts) Write down, but do not evaluate, a triple nested integral in spherical coordinates representing the integral of the function $f(x, y, z) = x^2 + y^2$ on the domain D ; D is the part of the ball of radius 1 centered at $(0, 1, 0)$ with $x \leq 0$ and $z \geq 0$.



ρ_2 is determined by eq. of sphere:

$$x^2 + (y-1)^2 + z^2 = 1$$

$$x^2 + y^2 + z^2 - 2y + 1 = 1$$

$$\rho^2 - 2\rho \sin\phi \sin\theta = 0$$

$$\rho_2 = 2 \sin\phi \sin\theta$$

And $f = x^2 + y^2$
 $= \rho^2 \sin^2\phi$

Then

$$\iiint_D f \, dV = \int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^{2 \sin\phi \sin\theta} (\rho^2 \sin^2\phi) (\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta)$$

$$= \int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^{2 \sin\phi \sin\theta} \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$$