EXAM 1

Math 212, 2013 Summer Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. ___________
2. ___________
3. ___________
4. ___________
5. ___________
6. ___________
7. ___________

Total Score ___________ (/100 points)

1. *(10 pts)* Suppose that $\vec{v}$ and $\vec{w}$ are nonzero vectors that are not parallel. Use the determinant to show that $\vec{v} \times \vec{w}$, $\vec{v}$, $\vec{w}$ is in right-hand order.

Explain why this argument relies on $\vec{v}$ and $\vec{w}$ being nonzero vectors that are not parallel.

$$\det\begin{pmatrix} \text{---}\ \vec{v}\times\vec{w}\ \text{---} \\ \text{---}\ \vec{v}\ \text{---} \\ \text{---}\ \vec{w}\ \text{---} \end{pmatrix} = (\vec{v}\times\vec{w}) \cdot \left((\vec{v})\times(\vec{w})\right) = \|\vec{v}\times\vec{w}\|^2$$

It is because of the assumptions that we know $\vec{v}\times\vec{w} \neq \vec{0}$, so the above determinant is positive.

Thus the listing is in R.H.O.

2. *(15 pts)* Find the symmetric equations for the line that is the intersection of the planes $3x - 2y + 6z = 7$ and $2x + y + z = 4$. *(Hint:* $(1, 1, 1)$ *might be useful in some way.)*

$\vec{n}_1 = \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix}$ and $\vec{n}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ are both $\perp$ to the direction vector for the line, so we use

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{pmatrix} -8 \\ 9 \\ 7 \end{pmatrix}$$

By inspection $\vec{X}_0 = (1,1,1)$ is on both planes, and thus the line.

Then
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -8 \\ 9 \\ 7 \end{pmatrix}$$

and the symmetric equations are

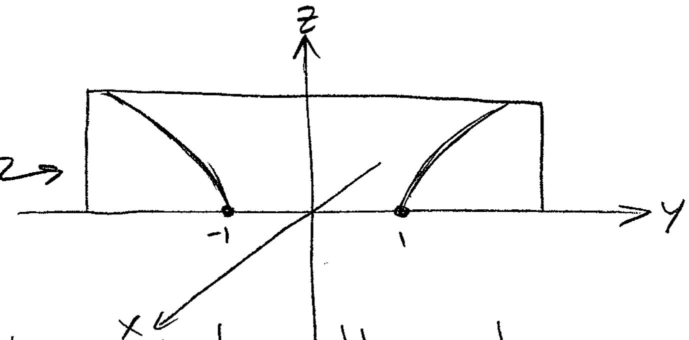
$$\boxed{\frac{x-1}{-8} = \frac{y-1}{9} = \frac{z-1}{7}}$$

3. *(15 pts)* Find the equation of the surface S that is the unique (right circular) hyperboloid of two sheets such that:

(a) the vertices of S are at the points $(0, 3, 0)$ and $(0, 5, 0)$

(b) the cross section of S in the plane $y = 1$ is a circle of radius 1

$y^2 - (x^2+z^2) = 1$ is the rotation of this curve around the y-axis.



This makes a hyperboloid of two sheets with vertices at $(0,-1,0)$ and $(0,1,0)$.

We can place those vertices at the desired points by shifting in the (pos.) y-dir. by 4, making the equation

$$(y-4)^2 - (x^2+z^2) = 1$$

As is then, the cross-section in $y=1$ is

$$x^2+z^2 = 8$$

To make this circle have radius 1 instead of $\sqrt{8} = 2\sqrt{2}$, we "squish" in both x and z directions by $\sqrt{8}$, making the final equation

$$(y-4)^2 - \left((\sqrt{8}\,x)^2 + (\sqrt{8}\,z)^2\right) = 1$$

or

$$\boxed{(y-4)^2 - 8x^2 - 8z^2 = 1}$$

4. *(15 pts)* The surface S has equation $x^2y - ze^{2xy} = 3y^2$.

(a) Find a ~~function f~~ whose graph is S.

$$\Leftrightarrow \quad z = e^{-2xy}(x^2y - 3y^2)$$

This is the graph of $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$\boxed{f(x,y) = e^{-2xy}(x^2y - 3y^2)}$$

(b) Find a function g such that S is a level set of g.

$$\Leftrightarrow \quad x^2y - ze^{2xy} - 3y^2 = 0$$

This is the $g=0$ level set of $g : \mathbb{R}^3 \to \mathbb{R}^1$ def. by

$$\boxed{g(x,y,z) = x^2y - ze^{2xy} - 3y^2}$$

(c) Find a function h that parametrizes S.

Using the graph param. with (a), we parametrize S by $h : \mathbb{R}^2 \to \mathbb{R}^3$

$$\boxed{h(u,v) = \begin{pmatrix} u \\ v \\ e^{-2uv}(u^2v - 3v^2) \end{pmatrix}}$$

5. *(15 pts)* Compute the following limit.

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{\sec(xy)}\right)$$

This is a combination of continuous functions (note that $\sec(0) = 1 \neq 0$), so the function is continuous.

So $\lim = \frac{0^2+0^2}{\sec(0)} = \frac{0}{1} = \boxed{0}$

6. *(15 pts)* Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^7$ is a linear transformation represented by a matrix A, and you are given

$$T(3,2,4) = \vec{v} \quad \text{and} \quad T(0,1,2) = \vec{w}$$

Find the first column of A.

1st col of $A = T(1,0,0)$

$= T\left(\frac{(3,2,4) - 2(0,1,2)}{3}\right)$

$= \frac{T(3,2,4) - 2T(0,1,2)}{3}$

$= \boxed{\frac{\vec{v} - 2\vec{w}}{3}}$

7. *(15 pts)* Suppose that at the moment when $t = 3$, the parametric curve $\vec{x}$ has position $(3, 2, 5)$ and velocity $(1, 2, 0)$, and the parametric curve $\vec{y}$ has position $(1, 3, 1)$ and velocity $(0, 1, 1)$. Compute the rate of change of $f(t) = \vec{x}(t) \cdot \vec{y}(t)$, and the velocity of the parametric curve $\vec{z}(t) = \vec{x}(t) \times \vec{y}(t)$.

$$f' = (\vec{x} \cdot \vec{y})' = \vec{x}' \cdot \vec{y} + \vec{x} \cdot \vec{y}'$$

$$f'(3) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \boxed{14}$$

$$\vec{z}' = (\vec{x} \times \vec{y})' = \vec{x}' \times \vec{y} + \vec{x} \times \vec{y}'$$

$$\vec{z}'(3) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 \\ -4 \\ 4 \end{pmatrix}}$$