EXAM 1

Math 212, 2014 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

9. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score ____________ (/100 points)

1. *(12 pts)* Let $\vec{v} = (3, 5, 1)$ and $\vec{w} = (2, 1, 1)$.

(a) Compute $\vec{v} \cdot \vec{w}$ and $\text{comp}_{\vec{w}}(\vec{v})$.

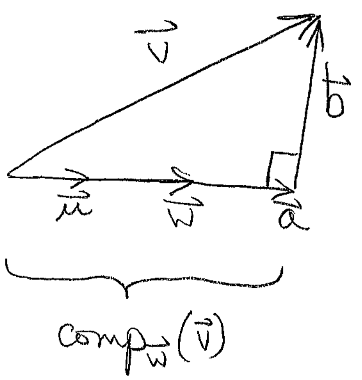
$$\vec{v} \cdot \vec{w} = 3 \cdot 2 + 5 \cdot 1 + 1 \cdot 1 = 12$$

$$\text{comp}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} = \frac{12}{\sqrt{6}} = 2\sqrt{6}$$

(b) Find the unit vector that points in the direction of the vector $\vec{w}$.

$$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{(2,1,1)}{\sqrt{6}}$$

(c) The vector $\vec{a}$ is parallel to $\vec{w}$, the vector $\vec{b}$ is orthogonal to $\vec{w}$, and $\vec{a} + \vec{b} = \vec{v}$. Find $\vec{a}$ and $\vec{b}$.
(Hint: The previous parts of this question should be useful.)



$$\vec{a} = \left(\text{comp}_{\vec{w}}(\vec{v})\right) \vec{u}$$
$$= \left(2\sqrt{6}\right)\left(\frac{(2,1,1)}{\sqrt{6}}\right)$$
$$= (4,2,2)$$

$$\vec{b} = \vec{v} - \vec{a} = (-1, 3, -1)$$

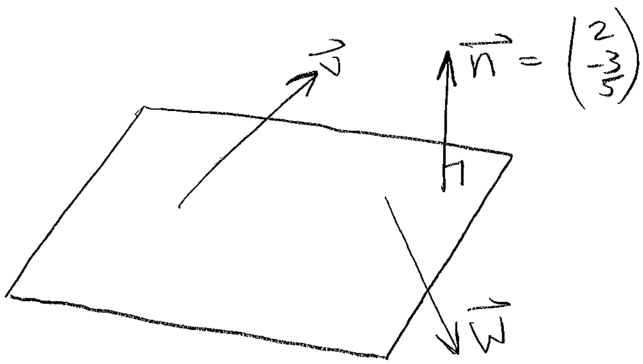
2. *(10 pts)* Show, as in class, that if $\vec{v} \times \vec{w} \neq \vec{0}$, then the list $\vec{v}$, $\vec{w}$, $\vec{v} \times \vec{w}$ is in right hand order.

$$\det\begin{pmatrix} \vec{v}\times\vec{w} \\ \vec{v} \\ \vec{w} \end{pmatrix} = (\vec{v}\times\vec{w}) \cdot \left((\vec{v}) \times (\vec{w})\right) = \|\vec{v}\times\vec{w}\|^2 > 0$$

So $\vec{v}\times\vec{w}, \vec{v}, \vec{w}$ is in RHO.

Cycling, we conclude also that $\vec{v}, \vec{w}, \vec{v}\times\vec{w}$ is in RHO.

3. *(10 pts)* The plane P has equation $2x - 3y + 5z = 3$; and $\vec{v} = (1, 2, 3)$. Find a parametrization of the line that is parallel to P, perpendicular to $\vec{v}$, and passes through the point $\vec{a} = (3, 4, 5)$.



Direction vector $\vec{w}$ is $\perp$ to both $\vec{n}, \vec{v}$; use $\vec{w} = \vec{n} \times \vec{v}$

$$= \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -19 \\ -1 \\ 7 \end{pmatrix}$$

Line param. by

$$\vec{x} = \vec{a} + t\vec{w} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} -19 \\ -1 \\ 7 \end{pmatrix}$$

4. *(10 pts)* Find the equation in spherical coordinates for the sphere of radius 13 centered at the point $(x, y, z) = (3, 4, 12)$.

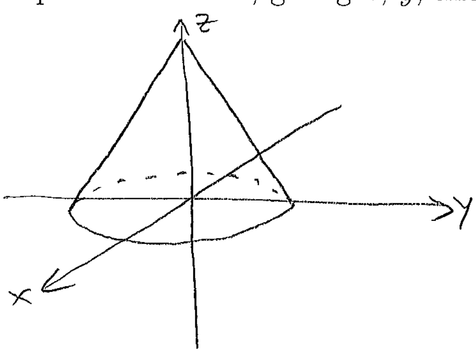
$$(x-3)^2 + (y-4)^2 + (z-12)^2 = 13^2$$

$$(x^2+y^2+z^2) - 6x - 8y - 24z + \cancel{(9+16+144)} = \cancel{169}$$

$$\rho^2 - 6\rho\sin\phi\cos\theta - 8\rho\sin\phi\sin\theta - 24\rho\cos\phi = 0$$

$$\rho = 6\sin\phi\cos\theta + 8\sin\phi\sin\theta + 24\cos\phi$$

5. *(12 pts)* The surface S is the part of the cone $z = 1 - r$ that is above the xy-plane. Find a parametrization, giving x, y, and z as functions of two other parameters.



$$x = r\cos\theta \qquad r \in [0,1]$$
$$y = r\sin\theta \qquad \theta \in [0, 2\pi]$$
$$z = 1-r$$

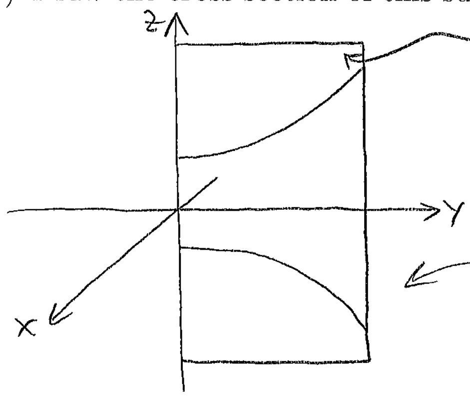
6. *(12 pts)* In this question we consider the solution set of $-x^2 - y^2 + z^2 = 1$.

(a) Around what line is the solution set rotationally symmetric?

$$-\underbrace{(x^2+y^2)} + z^2 = 1$$

x, y appear only in this form, so r.s. around z-axis.

(b) Draw the cross section of this surface in a corresponding half plane.

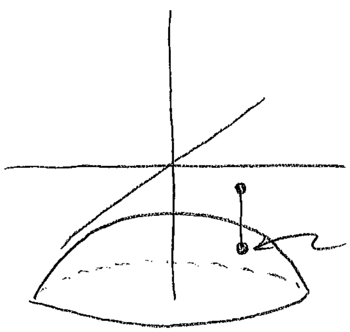


$x=0$, $y \geq 0$, so eq. becomes

$$-y^2+z^2=1$$

which describes this half hyperbola.

(c) Find a function f whose graph is the lower part of this surface. (Indicate clearly the domain and target for this function.)



$z=f(x,y)$

We choose the negative in

$$z = \pm\sqrt{1+x^2+y^2}$$

to get

$$f: \mathbb{R}^2 \to \mathbb{R}^1, \quad f(x,y) = -\sqrt{1+x^2+y^2}$$

7. *(12 pts)* Find a function g for which one of the level sets is the sphere of radius 4 centered at the point $(1, 2, 3)$. (Indicate clearly the domain and target for this function.)

Sphere has equation

$$(x-1)^2+(y-2)^2+(z-3)^2 = 16$$

This is the $g=16$ level set of

$$g: \mathbb{R}^3 \to \mathbb{R}^1, \quad g(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2$$

8. *(12 pts)* The vectors $\vec{v}_i$ of the left matrix in the product below are all orthogonal to each other, and have magnitudes 2, 3, and 5, respectively. Compute $\vec{a}_1 \cdot \vec{v}_3$.

$$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 8 & 7 & 9 \end{pmatrix} = \begin{pmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{pmatrix}$$

$$\vec{a}_1 = 1\vec{v}_1 + 2\vec{v}_2 + 8\vec{v}_3$$

$$\vec{a}_1 \cdot \vec{v}_3 = 1\underbrace{\vec{v}_1 \cdot \vec{v}_3}_{=0} + 2\underbrace{\vec{v}_2 \cdot \vec{v}_3}_{=0} + 8\vec{v}_3 \cdot \vec{v}_3$$

$$= 8\|\vec{v}_3\|^2$$

$$= 8 \cdot 25 = 200$$

9. *(10 pts)* A car starts at the location $(3, 5)$, and moves with velocity given by $(4\sin(2t), 6t\sin(t))$. Compute the location of the car at $t = \pi/4$.

$$\vec{v} = \begin{pmatrix} 4\sin(2t) \\ 6t\sin t \end{pmatrix} \qquad \vec{x} = \int \vec{v}\,dt = \begin{pmatrix} \int 4\sin 2t\,dt \\ \int 6t\sin t\,dt \end{pmatrix} + \vec{c}$$

$$\int 4\sin 2t\,dt = -2\cos 2t$$

$$\int \underbrace{6t}_{f}\underbrace{\sin t}_{g'}\,dt = (6t)(-\cos t) - \int (6)(-\cos t)\,dt$$

$$= -6t\cos t + 6\sin t$$

At $t=0$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \vec{c} \implies \vec{c} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

Then $\vec{x}(t) = \begin{pmatrix} -2\cos 2t \\ -6t\cos t + 6\sin t \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$\vec{x}(\pi/4) = \begin{pmatrix} 5 \\ 3\sqrt{2}\left(1 - \pi/4\right) + 5 \end{pmatrix}$$