EXAM 2

Math 212, 2014 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score ____________ (/100 points)

1. *(15 pts)* Compute – directly from the definition – the directional derivative $D_{\vec{v}}f(\vec{a})$ of the function given by $f(x,y,z) = x^2yz$ at the point $\vec{a} = (4,3,1)$ with velocity $\vec{v} = (1,1,2)$.

$$D_{\vec{v}}f(\vec{a}) = \frac{d}{dt}\Big|_{t=0} f(\vec{a}+t\vec{v}) = \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix}4+t\\3+t\\1+2t\end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0}\left((4+t)^2(3+t)(1+2t)\right) = \frac{d}{dt}\Big|_{t=0}\left((16+8t+t^2)(3+7t+2t^2)\right)$$

$$= \frac{d}{dt}\Big|_{t=0}\left(48+136t+91t^2+23t^3+2t^4\right)$$

$$= \left(0+136+182t+69t^2+8t^3\right)\Big|_{t=0}$$

$$= 136$$

2. *(15 pts)* Use the gradient vector to: (i) compute the directional derivative in the previous question; (ii) find the direction of fastest increase of f at $\vec{a}$; and (iii) find the maximum unit directional derivative of f at $\vec{a}$.

$$\nabla f = \begin{pmatrix}2xyz\\x^2z\\x^2y\end{pmatrix} \qquad \nabla f(\vec{a}) = \begin{pmatrix}24\\16\\48\end{pmatrix}$$

(i) $D_{\vec{v}}f(\vec{a}) = \nabla f(\vec{a})\cdot\vec{v} = \begin{pmatrix}24\\16\\48\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} = 136$

(ii) dir. of fastest increase $= \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{(24,16,48)}{\|(24,16,48)\|}$

$$= \frac{(3,2,6)}{\|(3,2,6)\|} = \frac{(3,2,6)}{7}$$

(iii) max udd. $= \|\nabla f\| = 8\|(3,2,6)\| = 8\cdot 7 = 56$

3. *(10 pts)* In a particular economic model, the variables r, K, L, T are used to compute an index I by the formula

$$I = rK^2 + 2KL + rLT^2$$

Suppose it is known that at a given moment r and L are not changing, that $dK/dt = 3$, and that $dT/dt = -2$. Find an expression for dI/dt in terms of the above variables.

$$\frac{dI}{dt} = \frac{\partial I}{\partial r}\underbrace{\frac{dr}{dt}}_{0} + \frac{\partial I}{\partial K}\underbrace{\frac{dK}{dt}}_{3} + \frac{\partial I}{\partial L}\underbrace{\frac{dL}{dt}}_{0} + \frac{\partial I}{\partial T}\underbrace{\frac{dT}{dt}}_{-2}$$

$$= (3)(2rK + 2L) + (-2)(2rLT)$$

$$= 6rK + 6L - 4rLT$$

4. *(10 pts)* Related to the above model is a consideration of the case where the index I is maintained as a constant. In this case, what is the condition on the given variables that indicates when K can be viewed as a function of the other variables? And, when that condition is satisfied, what is the expression for $\partial K/\partial L$?

$$rK^2 + 2KL + rLT^2 = F(r, K, L, T) = c$$

$$\frac{\partial F}{\partial K} = 2rK + 2L$$ ← K can be viewed as a fn of r, L, T if this is <u>not zero</u>.

In these cases,

$$\frac{\partial K}{\partial L} = -\frac{\partial F/\partial L}{\partial F/\partial K} = -\frac{2K + rT^2}{2rK + 2L}$$

5. *(10 pts)* The differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g(u, v) = (g_1, g_2)$. It is known that

$$\begin{aligned} D_{\vec{e}_1} g(2,3) &= (5,2) \\ D_{\vec{e}_2} g(2,3) &= (3,1) \end{aligned}$$

Concerning the function g at the point $(2, 3)$, find the Jacobian matrix, the gradient vectors of its component functions g_1 and g_2, and the unit directional derivative in the direction of the vector $(5, 7)$.

$D_{\vec{e}_1} = \frac{\partial}{\partial u}$, so $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is 1st col. of J_g

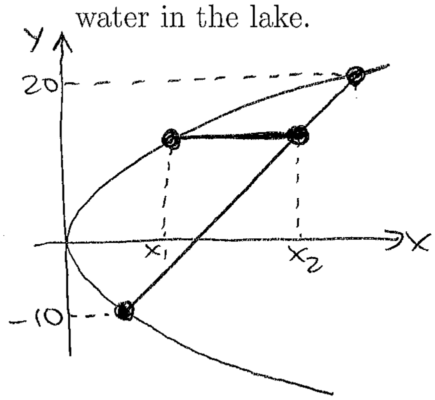
$D_{\vec{e}_2} = \frac{\partial}{\partial v}$, so $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is 2nd col. of J_g

Then $J_g = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$, so $\nabla g_1 = \text{1st row} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$\nabla g_2 = \text{2nd row} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\text{Unit dir. der.} = J_g \left(\frac{(5,7)}{\|(5,7)\|} \right) = \frac{\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}}{\sqrt{74}} = \begin{pmatrix} 46 \\ 17 \end{pmatrix} \Big/ \sqrt{74}$$

6. *(10 pts)* A lake is bounded by the curves $10x = y^2$ and $y = x - 20$ (distances measured in meters), and the depth of the lake at a point (x, y) is given by $d(x, y) = 20 + y$. Compute the volume of water in the lake.



$$V = \iint d \, dA = \int_{-10}^{20} \int_{y^2/10}^{y+20} (20 + y) \, dx \, dy$$

$$= \int_{-10}^{20} \left(20x + xy \right]_{x = y^2/10}^{x = y+20} dy$$

$$= \int_{-10}^{20} \left(20y + 400 + y^2 + 20y \right) - \left(2y^2 + \tfrac{1}{10} y^3 \right) dy$$

$$= \int_{-10}^{20} -\tfrac{1}{10} y^3 - y^2 + 40y + 400 \, dy$$

$$= \left(-\tfrac{1}{40} y^4 - \tfrac{1}{3} y^3 + 20y^2 + 400y \right]_{-10}^{20}$$

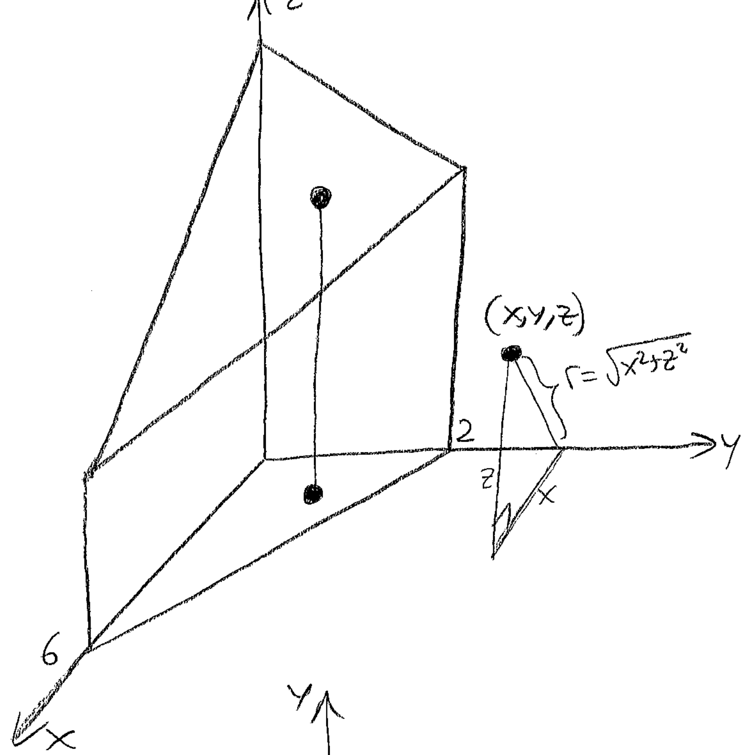
$$= \left(-4000 - \frac{8000}{3} + 8000 + 8000 \right) - \left(-250 + \frac{1000}{3} + 2000 - 40{,}000 \right)$$

$= 47{,}250$

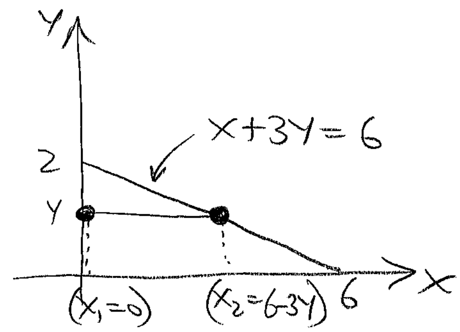
7. *(15 pts)* Write a triple nested integral (but do not evaluate it) that represents the moment of inertia of the solid D around the y-axis, where D is in the first octant, is bounded by the coordinate planes and the planes with equations $x + 3y = 6$ and $x + y + z = 10$, and has density given by $\delta(x, y, z) = 4 + x$.

$$I = \iiint_D r^2\,dm = \iiint_D r^2 \delta\,dV$$

$$= \iiint_D (x^2+z^2)(4+x)\,dV$$



Choose to do x,y slicing on "outside"; consider proj. of D to xy-plane



y ranges $[0, 2]$

x then ranges $[0, 6-3y]$

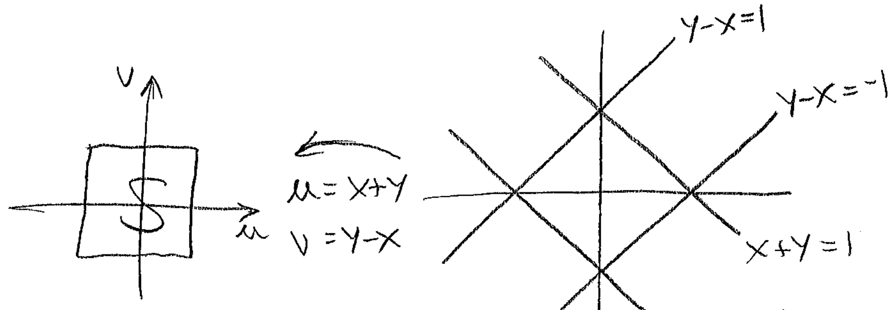
z then ranges $[0, 10-x-y]$

So,

$$I = \int_0^2 \int_0^{6-3y} \int_0^{10-x-y} (x^2+z^2)(4+x)\,dz\,dx\,dy$$

8. *(15 pts)* The domain D is a square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. Compute the integals below.

(a) $\iint_D (x+y)^4 \, dA$



$$= \iint_S u^4 \frac{\partial(x,y)}{\partial(u,v)} \, du\,dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$$

$$= \int_{-1}^{1}\int_{-1}^{1} u^4 \cdot \frac{1}{2} \, du\,dv$$

$$= \int_{-1}^{1} \left(\frac{1}{10} u^5 \Big]_{u=-1}^{u=1} \right) dv$$

$$= \int_{-1}^{1} \frac{1}{5} \, dv = \frac{2}{5}$$

(b) $\iint_D 4x^3 \, dA$

consider L = y-axis, $R(x,y) = (-x,y)$

D is symmetric over L

$f(R(x,y)) = f(-x,y) = 4(-x)^3 = -4x^3 = -f(x,y)$

so f has odd symm over L

Then $\iint = 0$ by symmetry.

(c) $\iint_D x + \sin^3(\pi(x+y)) \, dA \quad = \underbrace{\iint_D x \, dA} + \underbrace{\iint_D \sin^3(\pi(x+y))}$

on L_1 = y-axis,

D is symm.

$f(R(x,y)) = f(-x,y)$

$= (-x) = -x$

$= -f(x,y)$

So $\int = 0$ by symmetry

consider L_2 = line $y = -x$

D is symm over L_2

$f(R(x,y)) = f(-y,-x)$

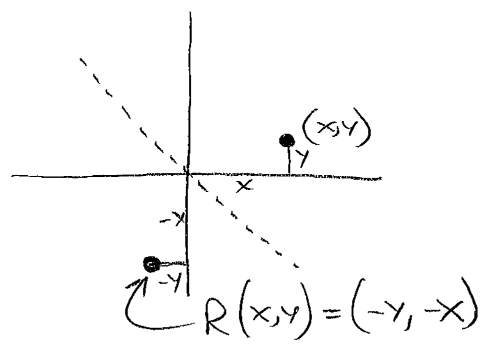
$= \sin^3(\pi((-y)+(-x)))$

$= \sin^3(-\pi(x+y))$

$= (-\sin(\pi(x+y)))^3$

$= -\sin^3(\pi(x+y))$

$= -f(x,y)$



So f is also odd over L_2

an $\iint = 0$ by symmetry

Original $\iint = 0 + 0 = 0$.