

# EXAM 1

Math 212, 2015 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING  
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

All answers must be simplified. All of the policies and guidelines  
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. \_\_\_\_\_

"I have adhered to the Duke Community  
Standard in completing this  
examination."

2. \_\_\_\_\_

3. \_\_\_\_\_

Signature: \_\_\_\_\_

4. \_\_\_\_\_

5. \_\_\_\_\_

6. \_\_\_\_\_

7. \_\_\_\_\_

8. \_\_\_\_\_

Total Score \_\_\_\_\_ (/100 points)

1. (15 pts) Find the area of the parallelogram defined by the vectors  $(3, 2, 5)$  and  $(1, -2, -1)$ .

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & 2 & 5 \\ 1 & -2 & -1 \end{vmatrix} = \begin{pmatrix} 8 \\ 8 \\ -8 \end{pmatrix}$$

$$\text{area} = \|\vec{v} \times \vec{w}\| = 8\sqrt{3}$$

2. (15 pts) Find the equation of the plane that perpendicularly bisects the line segment between  $(4, 1, 7)$  and  $(5, 4, 2)$ .

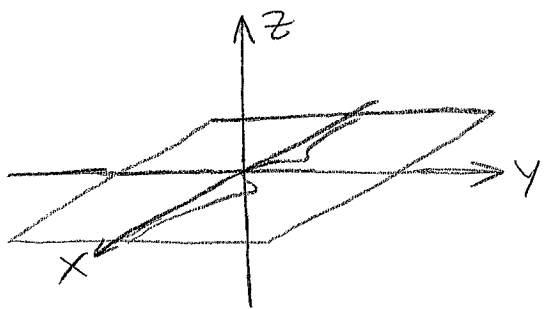
$$\vec{n} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$$

$$\vec{x}_0 = \frac{\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}}{2} = \begin{pmatrix} 9/2 \\ 5/2 \\ 9/2 \end{pmatrix}$$

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$x + 3y - 5z = \frac{-21}{2}$$

3. (10 pts) The curve  $C$  in the  $xy$ -plane is described by the equation  $y = x^4 e^{-x^2}$ . Find the equation of the surface  $S$  obtained by rotating  $C$  around the  $y$ -axis.



Eq of  $S$  is such that replacing  $\sqrt{x^2+z^2}$  with  $x$  results in  $y = x^4 e^{-x^2}$ ; and  $x, z$  appear only as part of " $\sqrt{x^2+z^2}$ ". This gives

$$y = (x^2+z^2)^2 e^{-(x^2+z^2)}$$

4. (10 pts) Find a function  $h$  whose graph is a level set of the function  $p: \mathbb{R}^3 \rightarrow \mathbb{R}^1$  defined by  $p(x, y, z) = x^2 y - x y^2 + z^3$ .

Level sets of  $p$  have equations

$$x^2 y - x y^2 + z^3 = c$$

This is equivalent to

$$z = (c - x^2 y + x y^2)^{1/3}$$

which is the graph of

$$h(x, y) = (c - x^2 y + x y^2)^{1/3}$$

5. (10 pts) Find a parametrization of the surface with equation  $x(y^2 + 3) - y^2 e^y = yz \cos^2 z$ .

This equation is equivalent to

$$x = \frac{y^2 e^y + yz \cos^2 z}{y^2 + 3}$$

Choosing  $y = u$ ,  $z = v$ , the resulting graph parametrization

is

$$\vec{X}(u, v) = \begin{pmatrix} \frac{u^2 e^u + uv \cos^2 v}{u^2 + 3} \\ u \\ v \end{pmatrix}$$

6. (10 pts) Compute the limit below.

$$\begin{aligned} & \lim_{\vec{x} \rightarrow \vec{0}} \frac{x^3 y - xy^2}{x^2 + y^2} \\ &= \lim_{r \rightarrow 0} \frac{(r \cos \theta)^3 (r \sin \theta) - (r \cos \theta) (r \sin \theta)^2}{r^2} \\ &= \lim_{r \rightarrow 0} r^2 \cos^3 \theta \sin \theta - r \cos \theta \sin^2 \theta \\ &= \lim_{r \rightarrow 0} (r^2 \cos^2 \theta \sin \theta) - \lim_{r \rightarrow 0} (r \cos \theta \sin^2 \theta) \quad \left( \begin{array}{l} \text{if these} \\ \text{exist} \end{array} \right) \end{aligned}$$

↗ both of these limits are zero because in each the first factor approaches zero and the second factor is bounded.

$$= 0 + 0 = 0$$

7. (15 pts) The linear transformations  $S$  and  $T$  are given below.

$$S(x, y, z) = \begin{pmatrix} 3x - 2y + z \\ 2x - 2y \\ y + 3z \end{pmatrix} \quad \text{and} \quad T(x, y, z) = \begin{pmatrix} x - y - z \\ 3x + 2y + z \\ x - z \end{pmatrix}$$

Find the matrix representing the linear transformation  $S \circ T$ , *without* explicitly composing these transformations.

$$S(\vec{e}_1) = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad S(\vec{e}_2) = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \quad S(\vec{e}_3) = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\text{So } S \text{ is represented by } B = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$$T(\vec{e}_1) = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad T(\vec{e}_2) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \quad T(\vec{e}_3) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{So } T \text{ is represented by } A = \begin{pmatrix} 1 & -1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Then  $S \circ T$  is represented by

$$BA = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -7 & -6 \\ -4 & -6 & -4 \\ 6 & 2 & -2 \end{pmatrix}$$

8. (15 pts) The velocity of a particle in the plane is  $\vec{v}(t) = (4e^{2t}, 3\sin t)$ , and its initial position is  $(5, 6)$ . Find the position of the particle as a function of time.

$$\begin{aligned}\vec{x}(t) &= \int \vec{v}(t) dt = \int \begin{pmatrix} 4e^{2t} \\ 3\sin t \end{pmatrix} dt \\ &= \begin{pmatrix} 2e^{2t} \\ -3\cos t \end{pmatrix} + \vec{c}\end{aligned}$$

At  $t=0$ ,  $\vec{x} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ , so

$$\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \vec{c} \quad \Rightarrow \quad \vec{c} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

Then

$$\vec{x}(t) = \begin{pmatrix} 2e^{2t} + 3 \\ -3\cos t + 9 \end{pmatrix}$$