EXAM 2

Math 212, 2015 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

Total Score ____________ (/100 points)

1. *(15 pts)* In this question we consider the function defined by $f(x,y) = x^2e^y - \sin(\pi xy)$.

(a) Compute the directional derivatives below, by any means from this course.

$$D_{\vec{e}_1} f(x,y) \quad \text{and} \quad D_{\vec{e}_2} f(x,y)$$

$$D_{\vec{e}_1} f = \frac{\partial f}{\partial x} = 2xe^y - \pi y \cos(\pi xy)$$

$$D_{\vec{e}_2} f = \frac{\partial f}{\partial y} = x^2 e^y - \pi x \cos(\pi xy)$$

(b) Is f differentiable? Explain your reasoning fully.

Both partials above are continuous.

$\Rightarrow$ f is continuously differentiable

$\Rightarrow$ f is differentiable

(c) Use parts (a) and (b) above (and note specifically where you use them!) to compute the directional derivative $D_{\vec{v}} f(\vec{a})$, where $\vec{v} = (4,2)$ and $\vec{a} = (1,0)$.

Because f is differentiable (part(b)!), we can compute with

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

$$= \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} \cdot \vec{v}$$

(using results of (a)!)

$$= \begin{pmatrix} 2 \\ 1-\pi \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \boxed{10 - 2\pi}$$

2. *(10 pts)* The variables w and z are functions of x and y, and x and y are the usual polar functions of r and θ (with $r > 0$). Simplify the expression below.

$$\frac{\partial}{\partial \theta}\left(x\, w_x\right)$$

$$\begin{array}{c} r \\ \theta \end{array} \longrightarrow \begin{array}{c} x \\ y \end{array} \longrightarrow \begin{array}{c} w \\ z \\ w_x \end{array}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta}(x w_x) &= \frac{\partial x}{\partial \theta} w_x + x \frac{\partial w_x}{\partial \theta} \\
&= -r\sin\theta\, w_x + x\left(\frac{\partial w_x}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w_x}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= -r\sin\theta\, w_x + r\cos\theta\left(w_{xx}(-r\sin\theta) + w_{xy}(r\cos\theta)\right) \\
&= -r\sin\theta\, w_x - r^2\sin\theta\cos\theta\, w_{xx} + r^2\cos^2\theta\, w_{xy}
\end{aligned}$$

3. *(10 pts)* We have $f(u,v) = (x,y)$, $g(x,y) = (s,t)$, and $h(s,t) = (p,q)$. We also know that, for certain values of u and v (and the resulting values of the other variables),

$$J_f = \begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad J_g = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

Write an expression for $\dfrac{\partial q}{\partial v}$ in terms of the partial derivatives of the function h.

$$J_{h\circ g\circ f} = J_h J_g J_f$$

$$\begin{pmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial s} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \frac{\partial q}{\partial v} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \frac{\partial q}{\partial s} & \frac{\partial q}{\partial t} \end{pmatrix} \begin{pmatrix} \cdot & 32 \\ \cdot & 18 \end{pmatrix}$$

So $\dfrac{\partial q}{\partial v} = 32\dfrac{\partial q}{\partial s} + 18\dfrac{\partial q}{\partial t}$

4. *(15 pts)* Use the definition of directional derivative (do NOT use partial derivatives or the gradient vector or equivalent) to find the slope of the tangent line to the graph of $f(x, y) = x^2y$ at the point with $x = 1$, $y = 2$, in the direction indicated by the vector $\vec{v} = (3, 4)$.

slope $= D_{\vec{u}} f(\vec{a})$, where $\vec{a} = (1,2)$, $\vec{u} = \frac{\vec{v}}{5}$

$$= \frac{1}{5} D_{\vec{v}} f(\vec{a})$$

$$= \frac{1}{5} \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 1+3t \\ 2+4t \end{pmatrix}$$

$$= \frac{1}{5} \frac{d}{dt}\Big|_{t=0} (1+3t)^2(2+4t)$$

$$= \frac{1}{5} \frac{d}{dt}\Big|_{t=0} (2 + 16t + 42t^2 + 36t^3)$$

$$= \frac{1}{5}(16)$$

$$= \boxed{\frac{16}{5}}$$

5. *(15 pts)* Use the gradient vector to find the equation of the tangent plane to the graph in the previous question at the indicated point.

Graph of f has eqn $z = x^2y$

which is equivalent to $x^2y - z = 0$

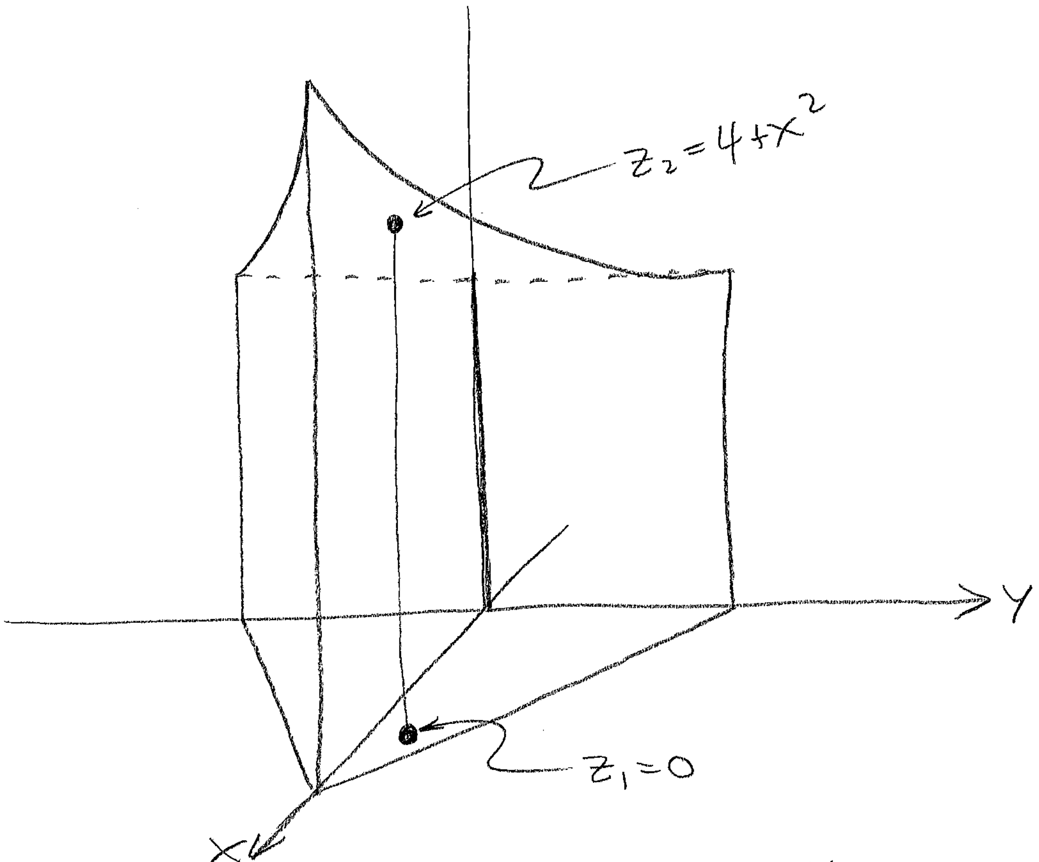
which is a level set of $g(x,y,z) = x^2y - z$

$\nabla g = \begin{pmatrix} 2xy \\ x^2 \\ -1 \end{pmatrix}$ is $\perp$ to surface; at $\vec{X_0} = (1,2,2)$,

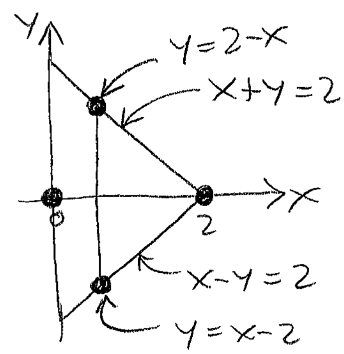
this gives us $\vec{n} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$.

Eq. of plane is $\boxed{4x + y - z = 4}$

6. *(15 pts)* The solid R is bounded by the surfaces $x+y=2$, $x-y=2$, $x=0$, $z=0$, and $z=4+x^2$. Set up, but do not evaluate, a (single) triple iterated integral representing $\iiint_R f(x,y,z)\,dV$



Projection to xy-plane is →



x-slices : $x_1=0$, $x_1=2$

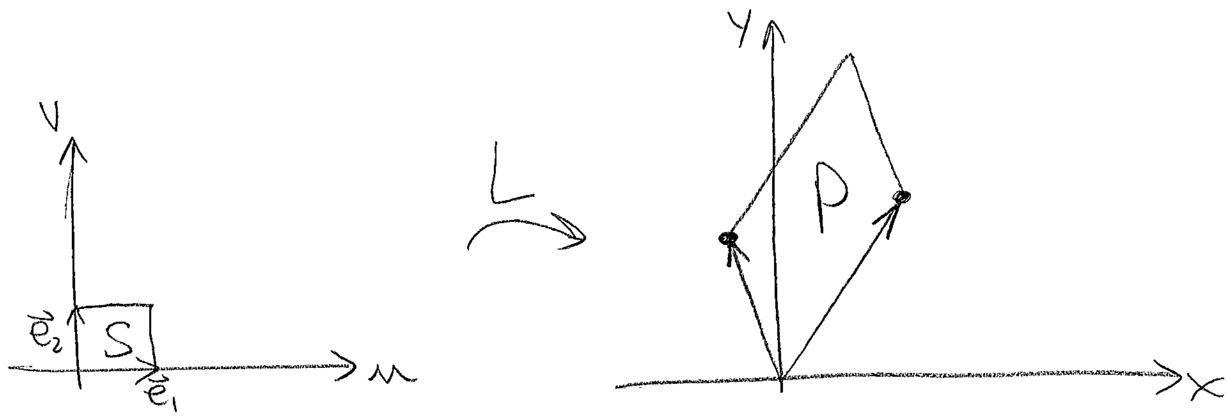
y-slices : $y_1=x-2$, $y_2=2-x$

Then as in top figure,

z-slices : $z_1=0$, $z_2=4+x^2$

So
$$\boxed{\iiint f\,dV = \int_0^2 \int_{x-2}^{2-x} \int_0^{4+x^2} f(x,y,z)\,dz\,dy\,dx}$$

7. *(20 pts)* The parallelogram P in $\mathbb{R}^2$ has edge vectors $(3,5)$ and $(-1,4)$. Set up, but do not evaluate, a (single) double iterated integral to compute $\iint_P x\,dx\,dy$. *(Hint: Use a linear transformation L, with $L(\vec{e}_1) = (3,5)$ and $L(\vec{e}_2) = (-1,4)$, to relate P to the unit square.)*



$$L(\vec{e}_1) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad L(\vec{e}_2) = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \Rightarrow L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3u - v \\ 5u + 4v \end{pmatrix}$$

$$J_L = \begin{pmatrix} 3 & -1 \\ 5 & 4 \end{pmatrix} \qquad |\det J_L| = |17| = 17$$

Then $$\iint_P x\,dx\,dy = \iint_S (3u - v)(17)\,du\,dv$$

$$= \int_0^1 \int_0^1 (3u - v)(17)\,du\,dv$$