EXAM 1

Math 212, 2015 Summer Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

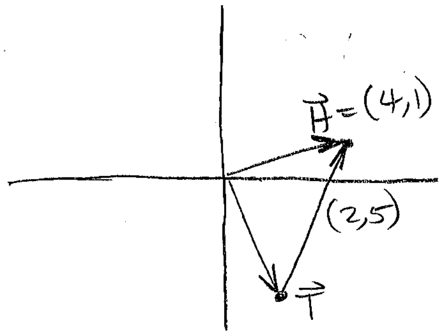
1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________
8. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score ____________ (/100 points)

1. (a) *(6 pts)* The vector $(2,5)$ is represented by an arrow whose head is at the point $(4,1)$. At what point is the tail of this arrow?



From the diagram,

$$\vec{H} = \vec{T} + (2,5)$$

So

$$\vec{T} = \vec{H} - (2,5) = (2,-4)$$

(b) *(8 pts)* Compute the angle between the vectors $(3,1,2)$ and $(1,2,3)$.

$$\|(3,1,2)\| = \sqrt{3^2+1^2+2^2} = \sqrt{14}$$

$$\|(1,2,3)\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$(3,1,2)\cdot(1,2,3) = 11$$

$$\theta = \arccos\left(\frac{(3,1,2)\cdot(1,2,3)}{\|(3,1,2)\|\,\|(1,2,3)\|}\right) = \arccos\left(\frac{11}{14}\right)$$

2. *(10 pts)* Bob is asked to compute the area of a parallelogram whose vertices are at the origin, $\vec{a} = (1,2,3)$, $\vec{b} = (4,5,6)$, and $\vec{c} = (5,7,9) = \vec{a} + \vec{b}$. Inadvertently he computes the magnitude of $\vec{a} \times \vec{c}$; and then upon noticing the second edge vector is $\vec{b}$, not $\vec{c}$, he computes the magnitude of $\vec{a} \times \vec{b}$. He is surprised to notice that he gets the same answer with both calculations.

Under what conditions on the vectors $\vec{a}$ and $\vec{b}$ defining a parallelogram, as above, will the first method above yield the correct answer? Justify your answer.

$$\vec{a}\times\vec{c} = \vec{a}\times(\vec{a}+\vec{b})$$

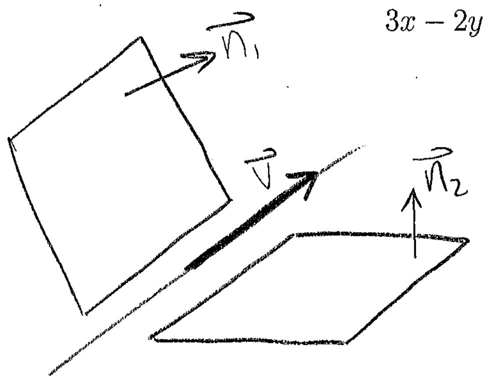
$$= \underbrace{\vec{a}\times\vec{a}}_{\substack{\text{this is } 0 \\ \text{because } \vec{a}\parallel\vec{a}}} + \vec{a}\times\vec{b} = \vec{a}\times\vec{b}$$

So not only do $\vec{a}\times\vec{c}$ and $\vec{a}\times\vec{b}$ have the same magnitudes, in fact they are the same vector.

There are <u>no</u> conditions.

3. *(13 pts)* Find the symmetric equations of the line that is parallel to both of the planes below, and which passes through the point $(3, 4, 5)$.

$$3x - 2y + z = 2 \quad \text{and} \quad x - 4z = 1$$



$$\vec{v} = \vec{n}_1 \times \vec{n}_2$$

$$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 8 \\ 13 \\ 2 \end{pmatrix}$$

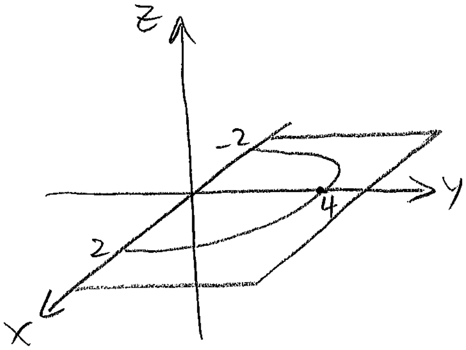
$$t = \boxed{\frac{x-3}{8} = \frac{y-4}{13} = \frac{z-5}{2}}$$

4. *(13 pts)* Give a clear geometric description of the surface with equation below. Be sure to note all relevant features of the surface.

$$x^2 + \sqrt{y^2 + z^2} = 4$$

This is rotationally symmetric around the x-axis.

In $z=0$, $y \geq 0$ half plane, $\sqrt{y^2+z^2} = y$, and we have



$$x^2 + y = 4$$

$$y = 4 - x^2$$

which in the half plane is the parabolic arc as shown.

The surface then is obtained by rotating this arc around the x-axis.

5. *(13 pts)* Find a function h for which one of the level sets is the surface with equation below.

$$x^5 - y + z^3 = 17$$

This equation is the $h=17$ level set of $h: \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$h(x,y,z) = x^5 - y + z^3$$

6. *(13 pts)* Find a parametrization of the curve described by $r = 3 + \sin^2(7\theta)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} (3+\sin^2(7\theta))\cos\theta \\ (3+\sin^2(7\theta))\sin\theta \end{pmatrix}$$

7. (a) *(3 pts)* State the full $\epsilon - \delta$ definition of the meaning of the statement

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

For every $\epsilon > 0$ there is a $\delta > 0$ so that

$$\left(0 < \|(x,y)-(a,b)\| < \delta\right) \Rightarrow \left(|f(x,y) - L| < \varepsilon\right)$$

(b) *(7 pts)* Your friend Bob believes (incorrectly) that the statement below is false.

$$\lim_{(x,y)\to(1,3)} (5y) = 15$$

In reference to the definition, he poses the value $\epsilon = 1$.
Choose a value of δ and show that with this delta you can satisfy the condition required in the definition for Bob's choice of ϵ.

Choose $\delta = 1/5$. Then

$$\|(x,y)-(1,3)\| < \delta$$

$$\Rightarrow \sqrt{(x-1)^2+(y-3)^2} < \frac{1}{5}$$

$$\rotatebox{90}{$\leq$}$$

$$\Rightarrow |y-3| < \frac{1}{5}$$

$$\Rightarrow |5y-15| < 1$$

$$\Rightarrow |f(x,y)-L| < \epsilon$$, as required

8. *(14 pts)* The line L is parametrized by $\vec{x}(t) = (3 - 2t, 2 + 5t)$, and the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T(x, y) = \begin{pmatrix} x + 2y \\ 3x - y \end{pmatrix}$$

Show that the image by T of the line L is also a line, and find a parametrization of that line.

$$\vec{x}(t) = \begin{pmatrix} 3-2t \\ 2+5t \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The image of L then is parametrized by

$$T(\vec{x}(t)) = T\begin{pmatrix} 3-2t \\ 2+5t \end{pmatrix}$$

$$= \begin{pmatrix} (3-2t) + 2(2+5t) \\ 3(3-2t) - (2+5t) \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 8t \\ 7 - 11t \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 7 \end{pmatrix} + t\begin{pmatrix} 8 \\ -11 \end{pmatrix}$$

This is a parametric line.