EXAM 1

Math 212, 2016 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

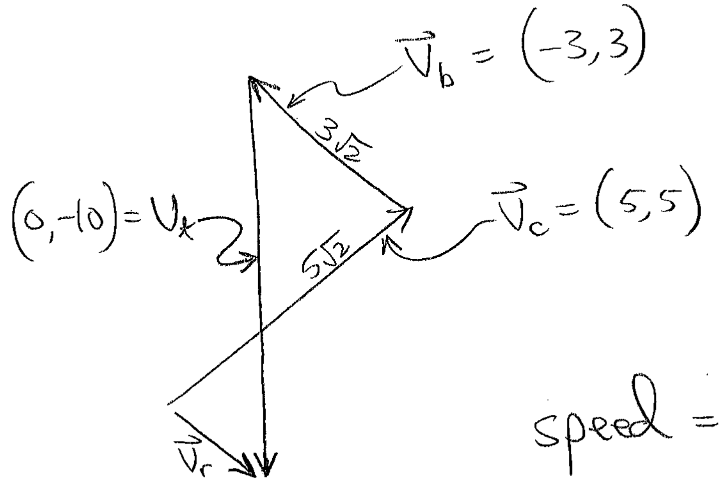
9. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score ____________ (/100 points)

1. *(6 pts)* The current in a river is flowing at $5\sqrt{2}$ feet per second directly northeast. Bob is in a boat whose velocity relative to the water (boat velocity minus water velocity) that is $3\sqrt{2}$ feet per second directly northwest. He then aims directly south and tosses a rock at 10 feet per second and hits a tree on the side of the river. What is the (horizontal) speed of the rock when it hits the tree?



$$\underset{(\text{rock})}{\vec{V}_r} = \underset{(\text{current})}{\vec{V}_c} + \underset{(\text{boat})}{\vec{V}_b} + \underset{(\text{toss})}{\vec{V}_t}$$

$$= \begin{pmatrix} 5 \\ 5 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\text{speed} = \|\vec{V}_r\| = \boxed{2\sqrt{2}}$$

2. *(8 pts)* Is the listing $(1,3,2)$, $(2,4,2)$, $(1,0,3)$ in right hand order, left hand order, or neither? And what is the volume of the parallelepiped defined by these three vectors?

$$\det\begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 0 & 3 \end{pmatrix} = (1)\det\begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} - (3)\det\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} + (2)\det\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} = -8$$

$\det < 0 \Rightarrow$ order is $\boxed{\text{left-hand}}$

$\text{vol.} = |\det| = \boxed{8}$

3. *(12 pts)* Derive the spherical coordinates equation for the sphere of radius 3 centered at $(3,0,0)$.

$$(x-3)^2 + y^2 + z^2 = 3^2$$

$$x^2 + y^2 + z^2 - 6x + 9 = 9$$

$$\rho^2 - 6\rho \sin\phi \cos\theta = 0$$

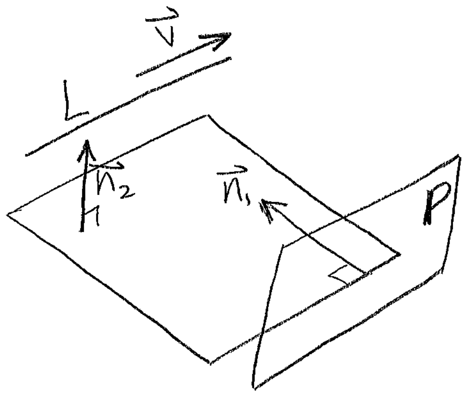
$$\boxed{\rho = 6 \sin\phi \cos\theta}$$

4. *(12 pts)* The line L has symmetric equations below.

$$\frac{x-3}{2} = y+1 = 2z-6 \;=t \;\Rightarrow\; \begin{aligned} x &= 2t+3 \\ y &= t-1 \\ z &= \tfrac{1}{2}t+3 \end{aligned}$$

and the plane P has equation $3x - 2y + z = 6$

Find the equation for the plane through the origin that is parallel to L and perpendicular to P.



Normal vector $\vec{n}_2$ is orthog. to both $\vec{v}$ and $\vec{n}_1$, so choose

$$\vec{n}_2 = \vec{v} \times \vec{n}_1 = \begin{pmatrix} 2 \\ 1 \\ 1/2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1/2 \\ -7 \end{pmatrix}$$

Eqn of plane is $\vec{n}_2 \cdot \vec{x} = \vec{n}_2 \cdot \vec{0}$

$$\boxed{2x - \tfrac{1}{2}y - 7z = 0}$$

5. *(13 pts)* The ellipsoid S with equation below can be obtained from the unit sphere by way of any of several sequences of geometric transformations. Identify such a sequence of transformations, including both the transformations and the order in which they are applied.

$$(3(x-2))^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1$$

$x^2 + y^2 + z^2 = 1$

$\downarrow$ "x" $\to$ "$3x$" $\leftarrow$ ("squish" in x-dir., factor of 3)

$(3x)^2 + y^2 + z^2 = 1$

$\downarrow$ "x" $\to$ "$x-2$" $\leftarrow$ (translation in x-dir, by +2)

$(3(x-2))^2 + y^2 + z^2 = 1$

$\downarrow$ "y" $\to$ "$\frac{y}{4}$" $\leftarrow$ ("stretch" in y-dir., factor of 4)

$(3(x-2))^2 + \left(\frac{y}{4}\right)^2 + z^2 = 1$

6. *(12 pts)* The surface S is parametrized by $\vec{x}(u,v) = (u-v, u+v, 6u-8v)$. Find the function f whose graph is S, and identify the domain and the target of f.

$$x = u - v \qquad x + y = 2u \qquad z = 3(2u) - 4(2v)$$
$$y = u + v \quad \Longrightarrow \quad y - x = 2v \quad \Longrightarrow \quad = 3(x+y) - 4(y-x)$$
$$z = 6u - 8v \qquad\qquad = 7x - y$$

$z = 7x - y$ is the graph of $f: \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x,y) = 7x - y$$

7. *(12 pts)* Compute the limit below or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^4}$$

Consider lines $y = mx$, parametrized by $(x,y) = (t, mt)$;

$$\lim_{t\to 0} \frac{(t^3)(mt)}{t^4 + (mt)^4} = \lim_{t\to 0} \frac{mt^4}{t^4 + m^4 t^4}$$

$$= \lim_{t\to 0} \frac{m}{1+m^4} = \frac{m}{1+m^4}$$

This gives different values for different lines, so the original limit does not exist.

8. *(13 pts)* $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and the information below is given:

$$T(\vec{v}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T(\vec{v}_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\vec{v}_1 - 2\vec{v}_2, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{v}_1 + \vec{v}_2$$

Find the matrix A that represents T.

$$T(\vec{e}_1) = T(3\vec{v}_1 - 2\vec{v}_2)$$

$$= 3T(\vec{v}_1) - 2T(\vec{v}_2)$$

$$= 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$$

$$T(\vec{e}_2) = T(\vec{v}_1 + \vec{v}_2)$$

$$= T(\vec{v}_1) + T(\vec{v}_2)$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{pmatrix} = \boxed{\begin{pmatrix} -5 & 5 \\ 0 & 5 \end{pmatrix}}$$

9. *(12 pts)* A particle is moving with position given by $\vec{x}(t) = (t, t^2 - 2t, 4t^2)$. Find the velocity and the acceleration of the particle at the point $(3, 3, 36)$.

$$\vec{v}(t) = (1, 2t-2, 8t)$$

$$\vec{a}(t) = (0, 2, 8)$$

$$(3,3,36) = (t, t^2-2t, 4t^2) \Rightarrow t=3$$

So at this point,

$$\boxed{\begin{aligned} \vec{v} &= (1, 4, 24) \\ \vec{a} &= (0, 2, 8) \end{aligned}}$$