EXAM 2

Math 212, 2016 Summer Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

Total Score ____________ (/100 points)

1. *(12 pts)* The surface S is the graph of $f(x,y) = x^2 e^{x^2-y}$. Without using differentiability, find the slope of the cross section to S at the point corresponding to $(2,4)$ in the domain, in the direction given by the vector $(5,12)$.

$\vec{v} = (5,12)$, $\quad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{13}$

$$\text{slope} = D_{\vec{u}} f(\vec{a})$$
$$= \frac{1}{13} D_{\vec{v}} f(\vec{a}) = \frac{1}{13} \frac{d}{dt}\Big|_{t=0} f(\vec{a}+t\vec{v}) = \frac{1}{13} \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 2+5t \\ 4+12t \end{pmatrix}$$
$$= \frac{1}{13} \frac{d}{dt}\Big|_{t=0} (2+5t)^2 e^{((2+5t)^2-(4+12t))}$$
$$= \frac{1}{13} \frac{d}{dt}\Big|_{t=0} (4+20t+25t^2) e^{(8t+25t^2)}$$
$$= \frac{1}{13}\left[(20+50t)e^{(8t+25t^2)} + (4+20t+25t^2)(8+50t)e^{8t+25t^2}\right]\Big|_{t=0}$$
$$= \frac{1}{13}[20+32] = 4$$

2. *(10 pts)* Bob is interested in computing the directional derivative $D_{\vec{v}} f(\vec{a})$. He is given that the function f is not directional linear at the point $\vec{a}$, but that it is directionally differentiable at that point.

He decides to try to use continuous differentiability to compute the directional derivative, asserting the statements below (some of which are true, some of which are not) as follows:

(a) The partial derivatives of f exist at the point $\vec{a}$.

(b) Those partial derivatives are continuous at that point.

(c) The function f is continuously differentiable at the point $\vec{a}$.

(d) The function f is differentiable at the point $\vec{a}$.

(e) The desired directional derivative can be computed as the product of the Jacobian matrix and the vector $\vec{v}$.

Is Bob's reasoning valid? If not, where is Bob's mistake? (That is, identify which of the above five statements is the FIRST false statement, and explain how you know it is false and how you know the previous statements are true.)

(a) is true, because every directionally differentiable function is partial differentiable.
(b) is false; because if the partials were continuous the function would be continuously differentiable and thus also directional linear, which it is not.
So Bob's reasoning is not valid.

3. *(12 pts)* Compute the directional derivative of the function $f(x,y) = y^3\sin(\pi x/4)$ at the point $\underbrace{(3,2)}_{\vec{a}}$ with velocity $\underbrace{(1,4)}_{\vec{v}}$.

(b/c f is diff'bl)

$$D_{\vec{v}}f(\vec{a}) \overset{!}{=} \nabla f(\vec{a})\cdot\vec{v}$$

$$= \begin{pmatrix} -\pi\sqrt{2} \\ 6\sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$= (24-\pi)\sqrt{2}$$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} \frac{\pi}{4}y^3\cos\left(\frac{\pi x}{4}\right) \\ 3y^2\sin\left(\frac{\pi x}{4}\right) \end{pmatrix}$$

$$\nabla f(\vec{a}) = \begin{pmatrix} \frac{\pi}{4}\cdot 8\cdot\frac{-\sqrt{2}}{2} \\ 3\cdot 4\cdot\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -\pi\sqrt{2} \\ 6\sqrt{2} \end{pmatrix}$$

4. *(15 pts)* Suppose that w is a differentiable function of x, y, and z, which are the usual spherical functions of ρ, ϕ, and θ. Simplify the expression below.

$$\frac{\partial}{\partial\phi}\left[y^2 w_z \rho\cos\theta\right]$$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$\begin{matrix} \rho & & x & & w \\ \phi & \rightarrow & y & \rightarrow & w_z \\ \theta & & z & & \end{matrix}$$

$$= \frac{\partial y^2}{\partial\phi} w_z\rho\cos\theta + y^2\frac{\partial w_z}{\partial\phi}\rho\cos\theta + y^2 w_z \underbrace{\frac{\partial(\rho\cos\theta)}{\partial\phi}}_{=0}$$

$$= 2y\frac{\partial y}{\partial\phi}w_z\rho\cos\theta + y^2\rho\cos\theta\left[\frac{\partial w_z}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial w_z}{\partial y}\frac{\partial y}{\partial\phi} + \frac{\partial w_z}{\partial z}\frac{\partial z}{\partial\phi}\right]$$

$$= 2y\rho\cos\phi\sin\theta\, w_z\rho\cos\theta + y^2\rho\cos\theta\left[w_{xz}\rho\cos\phi\cos\theta + w_{yz}\rho\cos\phi\sin\theta - w_{zz}\rho\sin\phi\right]$$

$$= y\rho^2\cos\theta\left[2w_z\cos\phi\sin\theta + yw_{xz}\cos\phi\cos\theta + yw_{yz}\cos\phi\sin\theta + yw_{zz}\sin\phi\right]$$

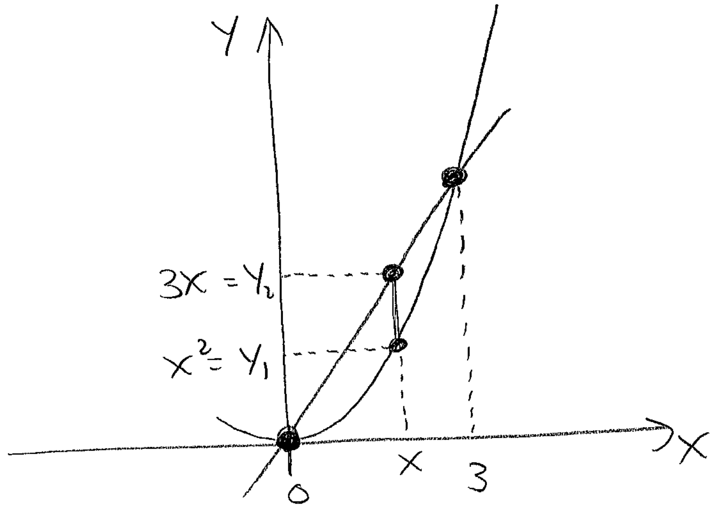
5. *(12 pts)* The region D in the xy-plane is bounded by the curves $y = 3x$ and $y = x^2$. The population density of bacteria in D is given by $\delta(x, y) = x + y$. Compote the total population of bacteria in D.

$$3x = x^2$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

$$x = 0, 3$$



$$\text{pop.} = \iint_D \delta \, dA$$

$$= \int_0^3 \int_{x^2}^{3x} (x+y) \, dy \, dx$$

$$= \int_0^3 \left(xy + \tfrac{1}{2}y^2 \right]_{y=x^2}^{y=3x} dx$$

$$= \int_0^3 \left(3x^2 + \tfrac{9}{2}x^2\right) - \left(x^3 + \tfrac{1}{2}x^4\right) dx$$

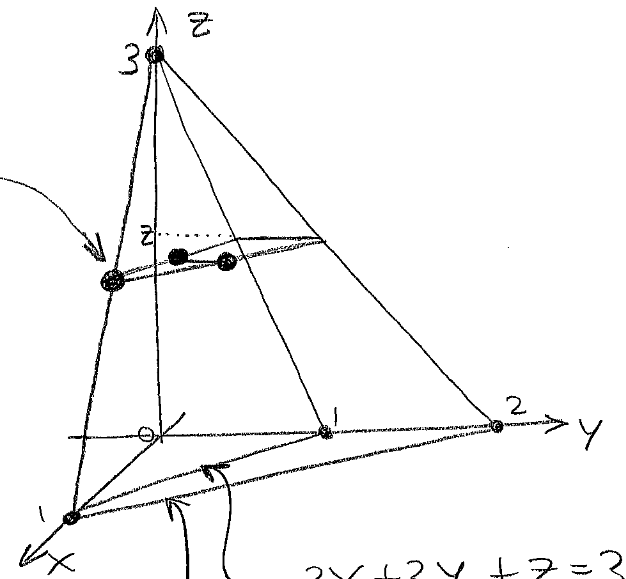
$$= \left(\tfrac{5}{2}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5 \right]_0^3$$

$$= \frac{135}{2} - \frac{81}{4} - \frac{243}{10}$$

$$= \frac{459}{20}$$

6. *(13 pts)* The solid tetrahedron T has vertices at $(1,0,0)$, $(0,1,0)$, $(0,2,0)$, and $(0,0,3)$, and its density function is $\delta(x,y,z) = e^{xyz}$.

Write out, but do not evaluate, a triple nested integral that represents the moment of inertia of T around the x-axis.



z goes from 0 to 3;

x starts at 0, ends at which is on these two planes, so $x_2 = \frac{3-z}{3}$;

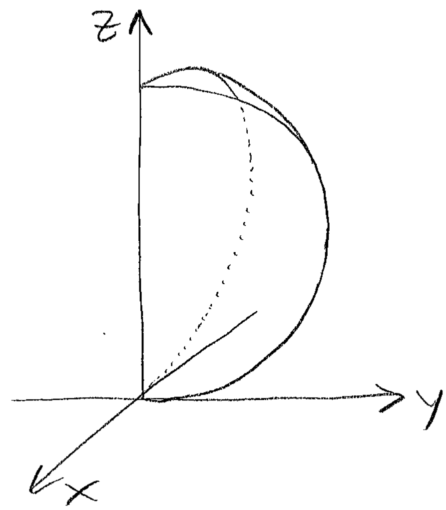
y starts on the back plane, ends on the front plane.

$3x+3y+z=3$

$6x+3y+2z=6$

$$I = \iiint r^2 \delta \, dV$$
$$= \int_0^3 \int_0^{\frac{3-z}{3}} \int_{\frac{3-z-3x}{3}}^{\frac{6-2z-6x}{3}} (y^2+z^2)(e^{xyz}) \, dy \, dx \, dz$$

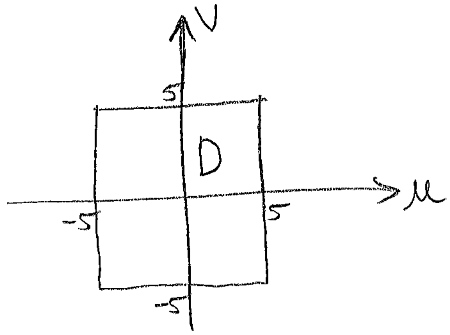
7. *(13 pts)* The ball B is centered at $(0,0,2)$ with radius 2. The solid S is the part of B with $y \geq 0$ and $x \leq 0$. Set up, but do not evaluate, a triple nested integral in spherical coordinates that represents the integral over S of the function $f(x,y,z) = \sin(xyz)$.



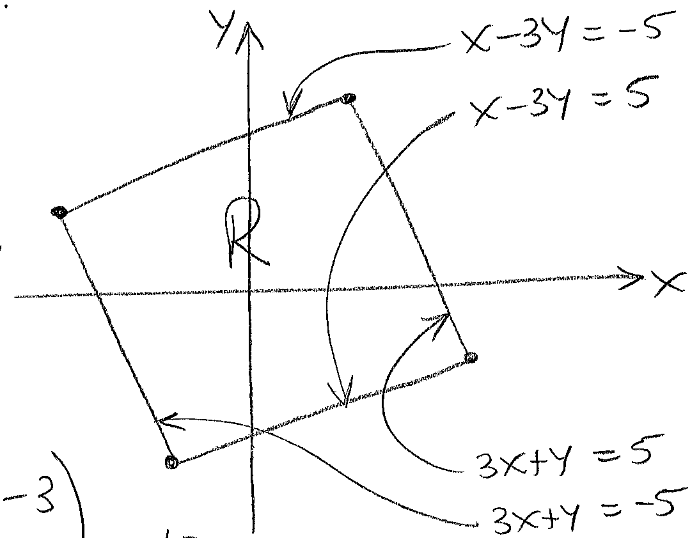
$$x^2+y^2+(z-2)^2 = 2^2$$
$$\rho^2 - 4z = 0$$
$$\rho^2 - 4\rho\cos\phi = 0$$
$$\rho = 4\cos\phi$$

$$\iiint f \, dV = \int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^{4\cos\phi} \sin\left(\rho^3 \sin^2\phi \cos\phi \sin\theta \cos\theta\right) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

8. *(13 pts)* The solid square R has vertices at $(1,2)$, $(-2,1)$, $(-1,-2)$, and $(2,-1)$. Compute the integral over R of the function $f(x,y) = 3x + y$.



$u = x - 3y$
$v = 3x + y$



$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} = 10$$

$$\iint_R 3x+y \, dx\,dy = \iint_D (3x+y) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_D (v)\left(\frac{1}{10}\right) du\,dv$$

Consider reflection over the u-axis, $R(u,v) = (u,-v)$

$f(u,v) = \frac{v}{10}$ is odd because

$$f(R(u,v)) = f(u,-v) = -\frac{v}{10} = -f(u,v)$$

D is symmetric.

So the integral is zero by symmetry.