EXAM 1

Math 212, 2016 Summer Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

Total Score ____________ (/100 points)

1. *(12 pts)* The component of the vector $\vec{x}$ in the direction of $(3,4)$ is 11, and the component of the vector $\vec{x}$ in the direction of $(-4,3)$ is 2. Find the vector $\vec{x}$.

$$\frac{\vec{X}\cdot(3,4)}{\|(3,4)\|} = 11 \Rightarrow 3X+4Y=55 \Rightarrow 12X+16Y=220$$

$$\frac{\vec{X}\cdot(-4,3)}{\|(-4,3)\|} = 2 \Rightarrow -4X+3Y=10 \Rightarrow -12X+9Y=30$$

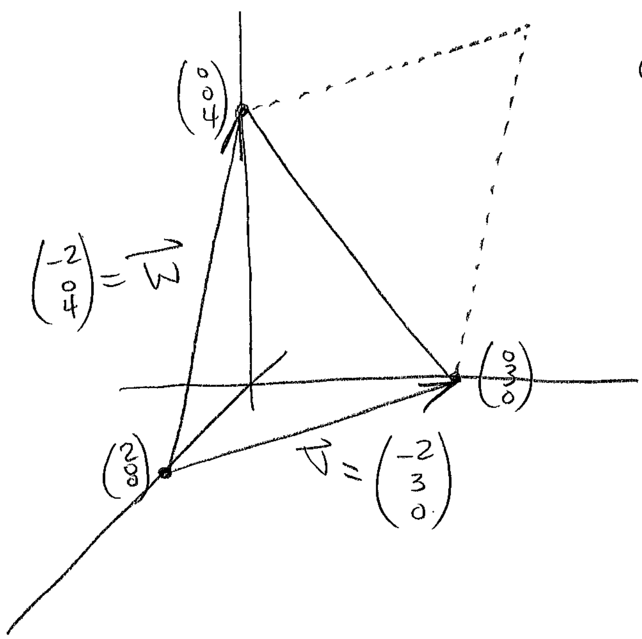
$$25Y = 250$$

$$\Downarrow$$

$$X=5 \Leftarrow Y=10$$

So $\vec{X} = (5,10)$

2. *(10 pts)* Compute the area of the triangle with vertices on the coordinate axes at 2, 3, and 4 respectively.



$$\text{area} = \frac{1}{2}\,\text{area}\left(\|(\vec{v},\vec{w})\right)$$

$$= \frac{1}{2}\|\vec{v}\times\vec{w}\|$$

$$= \frac{1}{2}\left\|\begin{pmatrix}-2\\3\\0\end{pmatrix}\times\begin{pmatrix}-2\\0\\4\end{pmatrix}\right\|$$

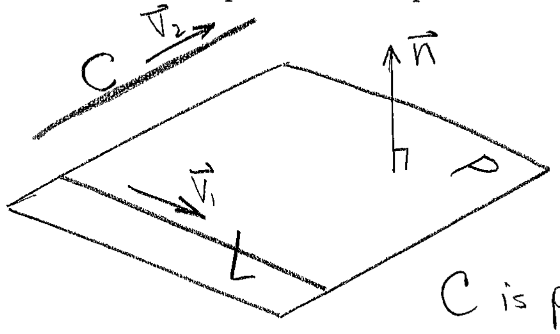
$$= \frac{1}{2}\left\|\begin{pmatrix}12\\8\\6\end{pmatrix}\right\|$$

$$= \sqrt{61}$$

3. *(15 pts)* The line L is parametrized by $(x, y, z) = (3-2t, 2+t, 4+3t)$, and the line C has symmetric equations

$$3x - 2 = 2y + 1 = z + 2$$

Find the equation of the plane P that contains L and is parallel to C.



L is param. by $\begin{pmatrix}3\\2\\4\end{pmatrix} + t\begin{pmatrix}-2\\1\\3\end{pmatrix}$

so we use $\vec{v}_1 = \begin{pmatrix}-2\\1\\3\end{pmatrix}$

C is param. by $3x-2 = 2y+1 = z+2 = t$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2/3\\-1/2\\-2\end{pmatrix} + t\begin{pmatrix}1/3\\1/2\\1\end{pmatrix}$

We can use

$\vec{n} = \vec{v}_1 \times \vec{v}_2$

So we use $\vec{v}_2 = \begin{pmatrix}1/3\\1/2\\1\end{pmatrix}$

$= \begin{pmatrix}-2\\1\\3\end{pmatrix} \times \begin{pmatrix}1/3\\1/2\\1\end{pmatrix} = \begin{pmatrix}-1/2\\3\\-4/3\end{pmatrix}$, & $\vec{x}_0 = \begin{pmatrix}3\\2\\4\end{pmatrix}$.

So the equation of the plane P is $\boxed{-\frac{1}{2}x + 3y - \frac{4}{3}z = \frac{-5}{6}}$

4. *(12 pts)* Give a clear geometric description of the surface S defined by the equation

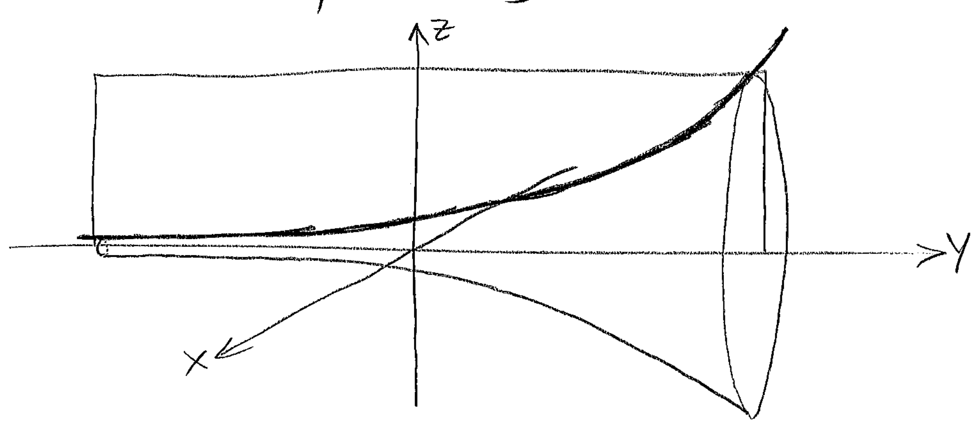
$$x^2 - e^{4y} + z^2 = 0$$

This is rotationally symmetric around the y-axis.

In $z \geq 0$ part of yz-plane $(x=0, z \geq 0)$, we have $\sqrt{x^2+z^2} = z$, so

$$z^2 = e^{4y} \Rightarrow z = e^{2y}$$

S is obtained by rotating this around the y-axis.



5. *(15 pts)* The surface P is a paraboloid with equation $(x-1)^2+(y-2)^2-z=3$.

(a) Find the function $h : \mathbb{R}^a \to \mathbb{R}^b$ whose graph is P, and identify the values of a and b.

$$\Leftrightarrow \quad z = (x-1)^2 + (y-2)^2 - 3$$

This is the graph of $h : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$h(x,y) = (x-1)^2 + (y-2)^2 - 3$$

(b) Find the function $q : \mathbb{R}^c \to \mathbb{R}^d$ for which one of the level sets is P, and identify the values of c and d.

$$(x-1)^2 + (y-2)^2 - z = 3$$

This is the level set $g^{-1}(3)$ of the function

$$g : \mathbb{R}^3 \to \mathbb{R}^1$$

given by

$$g(x,y,z) = (x-1)^2 + (y-2)^2 - z$$

6. *(10 pts)* Find a parametrization of the surface with equation below.

$$xe^{3y-z} + ye^z = 1 \iff x = e^{(z-3y)}(1-ye^z)$$

Letting $y = u$, $z = v$, and $x = x(y,z) \nearrow$, we get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{(v-3u)}(1-ue^v) \\ u \\ v \end{pmatrix}$$

7. *(13 pts)* Compute the limit below.

$$\lim_{\vec{x}\to\vec{0}} \frac{(x^3 - xy^2)e^{x^2+y^2}}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{(r\cos\theta)^3 - (r\cos\theta)(r\sin\theta)^2}{r^2} e^{r^2}$$

$$= \lim_{r\to 0} \underbrace{(re^{r^2})}_{\text{this}\to 0} \underbrace{(\cos^3\theta - \cos\theta\sin^2\theta)}_{\text{this is bounded}}$$

$$= 0$$

8. *(13 pts)* The lines L_1 and L_2 are parallel, and their images by the linear transformation T are the lines L_3 and L_4. Show that the lines L_3 and L_4 are parallel.

$L_1: \vec{x} = \vec{x}_1 + t\vec{v}$

$L_2: \vec{x} = \vec{x}_2 + t\vec{v}$

because L_1, L_2 parallel.

Then L_3 is parametrized by

$\vec{x} = T(\vec{x}_1 + t\vec{v}) = T(\vec{x}_1) + t\,T(\vec{v})$

and L_4 is parametrized by

$\vec{x} = T(\vec{x}_2 + t\vec{v}) = T(\vec{x}_2) + t\,T(\vec{v})$

So L_3, L_4 have the same direction vector, $T(\vec{v})$, and so L_3, L_4 are parallel.